# Fundamentals of Analog & Mixed Signal VLSI Design

## Noise in Circuits and Systems

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# Outline

- **Introduction**
- Random signals and noise
- Main noise sources of circuit components
- Noise models of basic components
- Noise calculation in continuous-time (CT) circuits
- Summary

# Different Types of Noise – Interference Noise

- Unwanted interaction between circuit and noise sources
- Noise sources not always easy to identify and localize
- Can be **random** or **deterministic**
- Examples: power supply noise, capacitive coupling, substrate coupling
- Can be **reduced** by **careful wiring**, **layout** and **grounding**
- Interference noise will **not be covered here**

David Johns and Ken Martin, *Analog Integrated Circuit Design*, 1st ed., Wiley 1997.

# Different Types of Noise – Inherent Noise

- Noise generated in **active** and **passive devices** (mainly resistors)
- **Random noise** due to **fluctuations** of physical quantities such as the carrier velocity
- Since source is random → it can be reduced but **NEVER eliminated**
- Examples: **thermal noise**, **shot noise**, and **flicker noise**
- Not strongly affected by wiring or layout
- Noise can be reduced by proper circuit design
- To design low-noise circuits requires understanding the parameters that determine the noise sources and how these noise sources propagate in the circuit
- Requires tools for analyzing and optimizing noise

📖 David Johns and Ken Martin, *Analog Integrated Circuit Design*, 1st ed., Wiley 1997.

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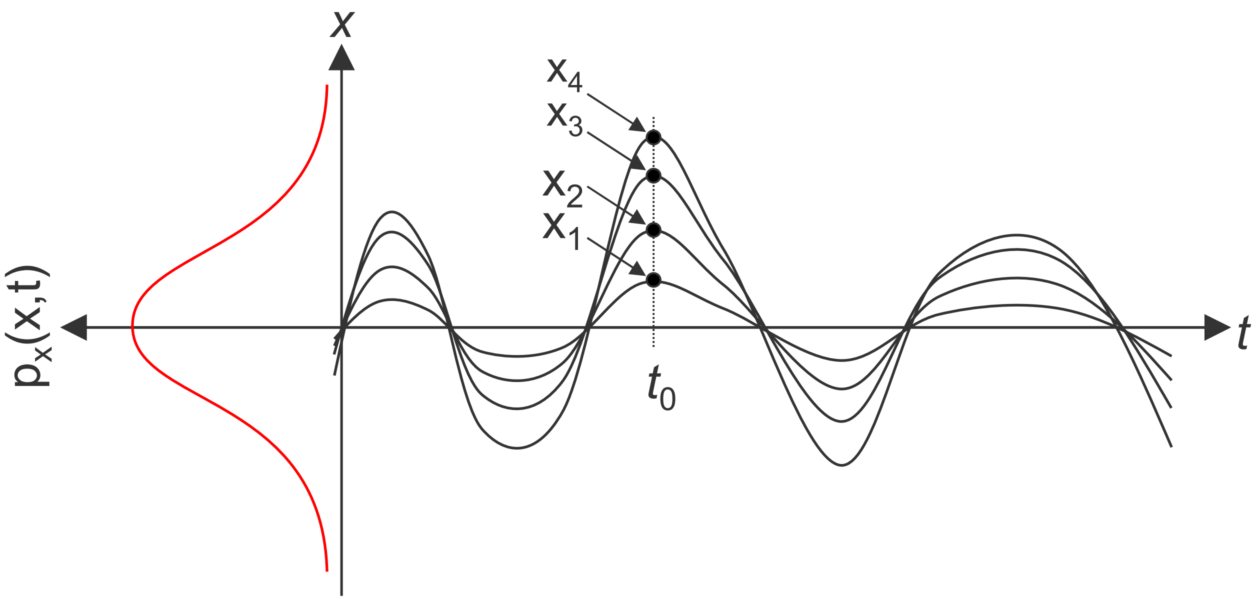
# Random Process – Definition




- The **fluctuation** of physical quantities can be mathematically modeled by a **random** or **stochastic process**
- A random process or stochastic process can be defined as a family of function $x(t)$, where $x(t = t_0)$ is a **random variable** which has a 1st-order distribution function $p_x(x, t)$
- The instantaneous amplitude of the signal $x(t)$ at time $t_0$ **cannot be predicted** since it is a random variable

# Random Process – Mean Value, Autocorrelation

- The **mean** or **expected value** $m_x(t)$ is obtained by averaging the amplitude $x$ at a given time $t$ over all possible values

$$m_x(t) = \mathrm{E}[x(t)] \triangleq \int_{-\infty}^{+\infty} x \cdot p_x(x,t) \cdot dx$$

- The **autocorrelation** function (ACF) evaluates the statistical dependency between two instantaneous values at time $t_1$ and $t_2$

$$R_x(t_1,t_2) = \mathrm{E}[x(t_1)x(t_2)] \triangleq \iint_{-\infty}^{+\infty} x_1 \cdot x_2 \cdot p_x(x_1,x_2,t_1,t_2) \cdot dx_1 dx_2$$

- where $x_1 = x(t_1)$, $x_2 = x(t_2)$ and $p_x(x_1,x_2,t_1,t_2)$ is the 2nd-order distribution function of process $x(t)$
- The **autocovariance** function is defined by

$$C_x(t_1,t_2) = E\big[\big(x(t_1) - m_x(t_1)\big)\big(x(t_2) - m_x(t_2)\big)\big] = R_x(t_1,t_2) - m_x(t_1) \cdot m_x(t_2)$$

A. Papoulis, *Probability, Random Variables and Stochastic Processes*, 1st ed., 1981.

# Random Process – Stationary Process

- The random process is called **stationary** in the wide sense if its **mean value is constant** and its autocorrelation function only depends on the **time difference** $\tau = t_2 - t_1$

$$m_x = \text{constant}$$
$$R_x(t_1, t_2) = R_x(\tau)$$
$$C_x(t_1, t_2) = C_x(\tau) = R_x(\tau) - m_x^2$$

- Note that $C_x(\tau) = R_x(\tau)$ for $m_x = 0$
- The normalized autocovariance function (or correlation coefficient) is defined as

$$\rho_x(\tau) = \frac{C_x(\tau)}{\sigma_x^2} \text{ where } \sigma_x^2 = C_x(0) \text{ is the } \textbf{variance}$$

- For **real** processes $R_x(\tau)$ and $C_x(\tau)$ are **even functions**

$$R_x(\tau) = R_x(-\tau) \text{ and } C_x(\tau) = C_x(-\tau)$$

- and have the following properties

$$C_x(0) = \sigma_x^2,\ R_x(0) = \sigma_x^2 - m_x^2 \text{ and } \rho_x(0) = 1$$

A. Papoulis, *Probability, Random Variables and Stochastic Processes*, 1st ed., 1981.

# Random Process – Ergodic Process

- A process is said **ergodic** when time average is equal to ensemble average

$$\bar{x} = \lim_{T\to\infty} \frac{1}{T} \cdot \int_0^T x(t) \cdot dt = E[x] = m_x$$

$$\overline{x^2} = \lim_{T\to\infty} \frac{1}{T} \cdot \int_0^T x^2(t) \cdot dt = E[x^2] = R_x(0)$$

A. Papoulis, *Probability, Random Variables and Stochastic Processes*, 1st ed., 1981.

# Random Process – Power Spectral Density (PSD)

- Noise corresponding to a **stationary** process $x(t)$ is often characterized in the frequency domain by the **Power Spectral Density** (PSD) which is the Fourier transform of its autocorrelation function

$$S_x(f) = \mathcal{F}\{R_x(\tau)\} = \int_{-\infty}^{+\infty} R_x(\tau) e^{-j2\pi f\tau} d\tau$$

$$R_x(\tau) = \mathcal{F}^{-1}\{S_x(f)\} = \int_{-\infty}^{+\infty} S_x(f) e^{+j2\pi f\tau} df$$

- PSD unit in [$V^2$/Hz] if $x(t)$ is a voltage or [$A^2$/Hz] if $x(t)$ is a current
- The above definition use the **bilateral Fourier transform (so watch out factor 2)!**
- Usually the unilateral PSD $S_x^+(f)$ is measured or simulated. It is related to the bilateral Fourier transform by

$$S_x^+(f) = \begin{cases} 0 & \text{for } f < 0 \\ S_x(f) & \text{for } f = 0 \\ 2S_x(f) & \text{for } f > 0 \end{cases}$$

A. Papoulis, *Probability, Random Variables and Stochastic Processes*, 1st ed., 1981.

# Random Process – Power and RMS Value

- The **total power** associated to the random process $x(t)$ is simply obtained by integrating its PSD over frequency

$$P_x = \int_{-\infty}^{+\infty} S_x(f)df = R_x(0)$$

- The total power corresponds to the square of the RMS value
- For a noise **voltage** PSD

$$V_{n,rms}^2 = \int_{-\infty}^{+\infty} S_{v_n}(f)df = \int_{0}^{+\infty} S_{v_n}^{+}(f)df$$

- For a noise **current** PSD

$$I_{n,rms}^2 = \int_{-\infty}^{+\infty} S_{i_n}(f)df = \int_{0}^{+\infty} S_{i_n}^{+}(f)df$$

A. Papoulis, *Probability, Random Variables and Stochastic Processes*, 1st ed., 1981.

# Random Process – Cross-correlation and Cross-PSD

- If we have two real jointly stationary processes $x(t)$ and $y(t)$, we can define their cross-correlation function (CCF) and cross-covariance by

$$R_{xy}(\tau) = E[x(t+\tau)y(t)]$$

$$C_{xy}(\tau) = E\left[(x(t+\tau) - m_x)\left(y(t+\tau) - m_y\right)\right] = R_{xy}(\tau) - m_x \cdot m_y$$

- The normalized cross-covariance is defined as

$$\rho_{xy}(\tau) = \frac{C_{xy}(\tau)}{\sigma_x \cdot \sigma_y}$$

- Two processes $x(t)$ and $y(t)$ are **uncorrelated** if $C_{xy}(\tau) = 0$
- The cross-PSD of the two real jointly stationary processes $x(t)$ and $y(t)$, is the Fourier transform of its cross-correlation function

$$S_{xy}(f) = \mathcal{F}\{R_{xy}(\tau)\} = \int_{-\infty}^{+\infty} R_{xy}(\tau) e^{-j2\pi f\tau} d\tau$$

A. Papoulis, *Probability, Random Variables and Stochastic Processes*, 1st ed., 1981.

# Noise Summation and Correlation

- If $z(t) = x(t) + y(t)$ is the sum of two **real stationary processes**, then

$$R_z(\tau) = R_x(\tau) + R_y(\tau) + R_{xy}(\tau) + R_{yx}(\tau)$$

- In case processes $x(t)$ and $y(t)$ are **uncorrelated**, then

$$C_{xy}(\tau) = C_{yx}(\tau) = 0 \implies R_{xy}(\tau) = R_{yx}(\tau) = m_x m_y \implies R_z(\tau) = R_x(\tau) + R_y(\tau) + 2m_x m_y$$

- If in addition $m_x = 0$ or $m_y = 0$ (or both), then

$$R_z(\tau) = R_x(\tau) + R_y(\tau)$$

- In this case the variance of $z(t)$ is given by

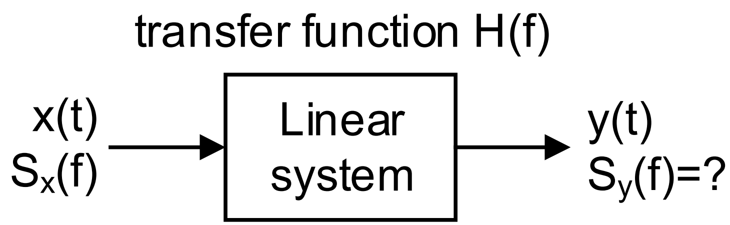
$$R_z(0) = \sigma_z^2 = R_x(0) + R_y(0) = \sigma_x^2 + \sigma_y^2$$

- In case process $x(t)$ and $y(t)$ are **correlated** and $m_x = 0$ or $m_y = 0$ (or both), then the variance is given by

$$\sigma_z^2 = \sigma_x^2 + \sigma_y^2 + 2c\sigma_x\sigma_y \text{ where } c = \rho_{xy}(0) = \rho_{yx}(0)$$

- where $c$ is the **correlation coefficient**

# PSD at the Output of a Linear System



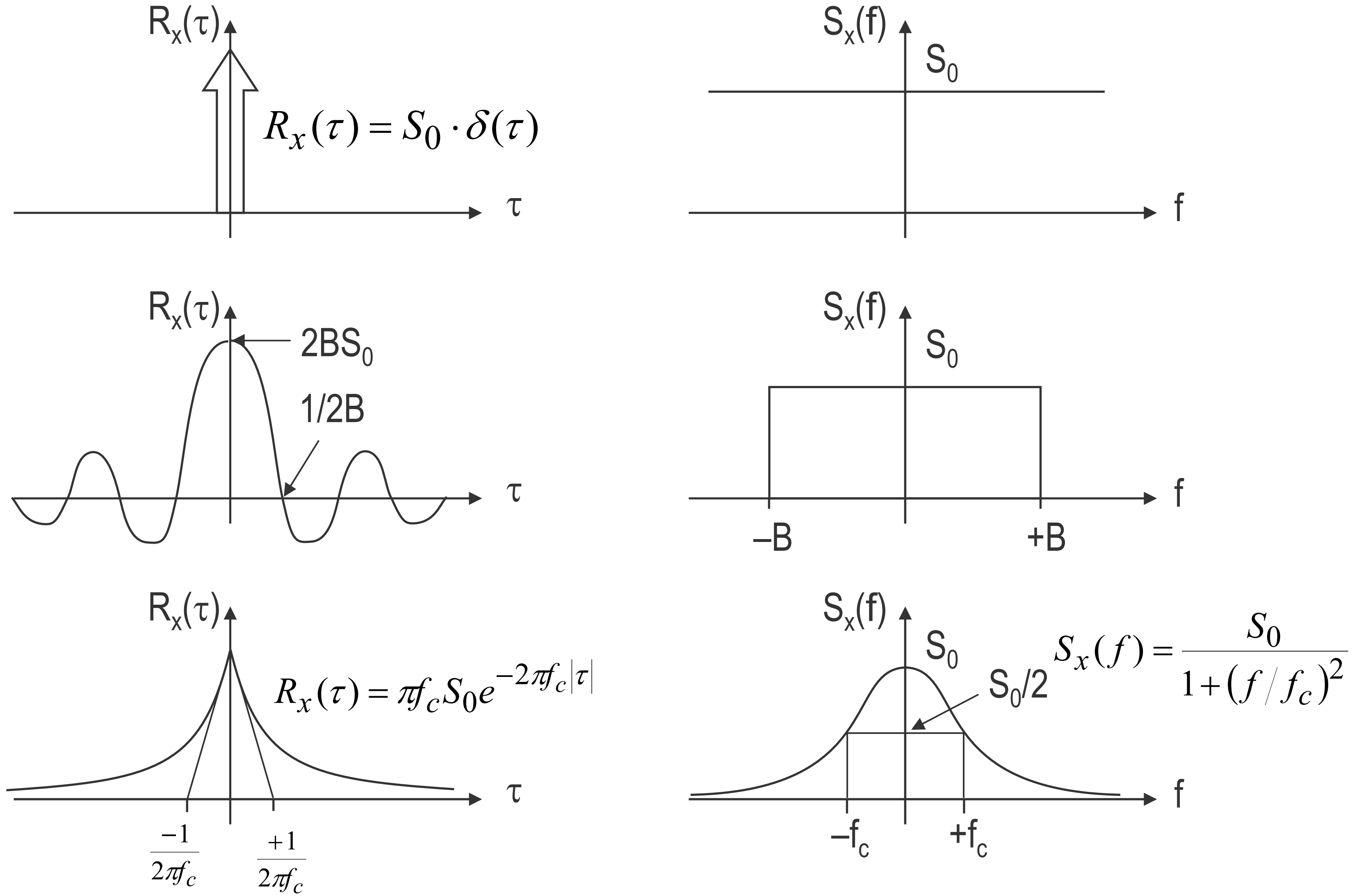

- Assuming the input is a **stationary random process** $x(t)$ having a PSD $S_x(f)$, the output $y(t)$ will also be a stationary random process having a PSD $S_y(f)$ given by

$$S_y(f) = |H(f)|^2 \cdot S_x(f)$$

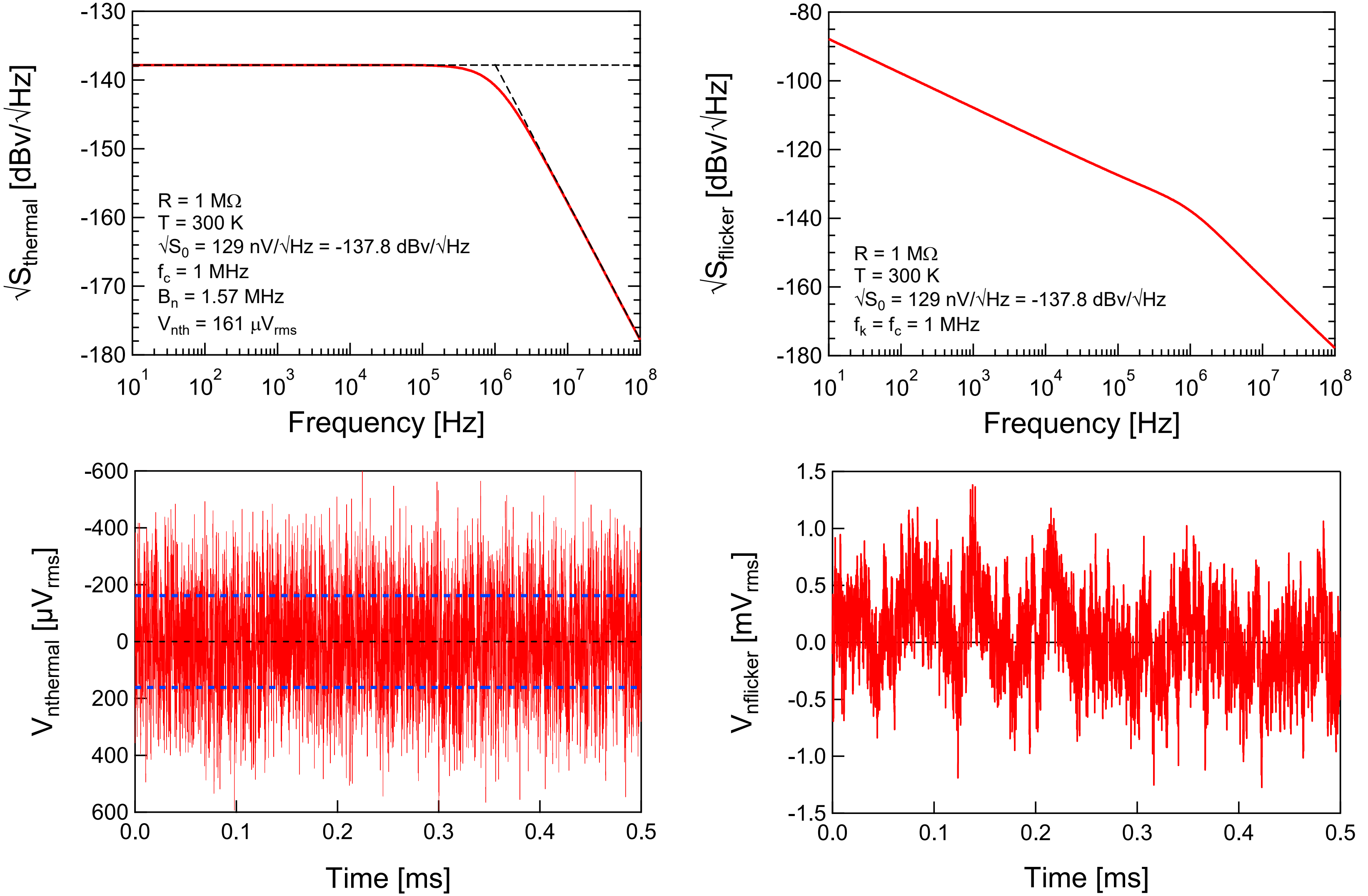
- The above relation is called the **Wiener-Kintchine theorem**
- The linear system could be a filter which shapes the input noise according to its square magnitude (phase does not play a role in this case)
- If the input is a white noise $S_x(f) = S_0 = cte$, the output PSD is then proportional to the square magnitude of the filter transfer function

A. Papoulis, *Probability, Random Variables and Stochastic Processes*, 1st ed., 1981.

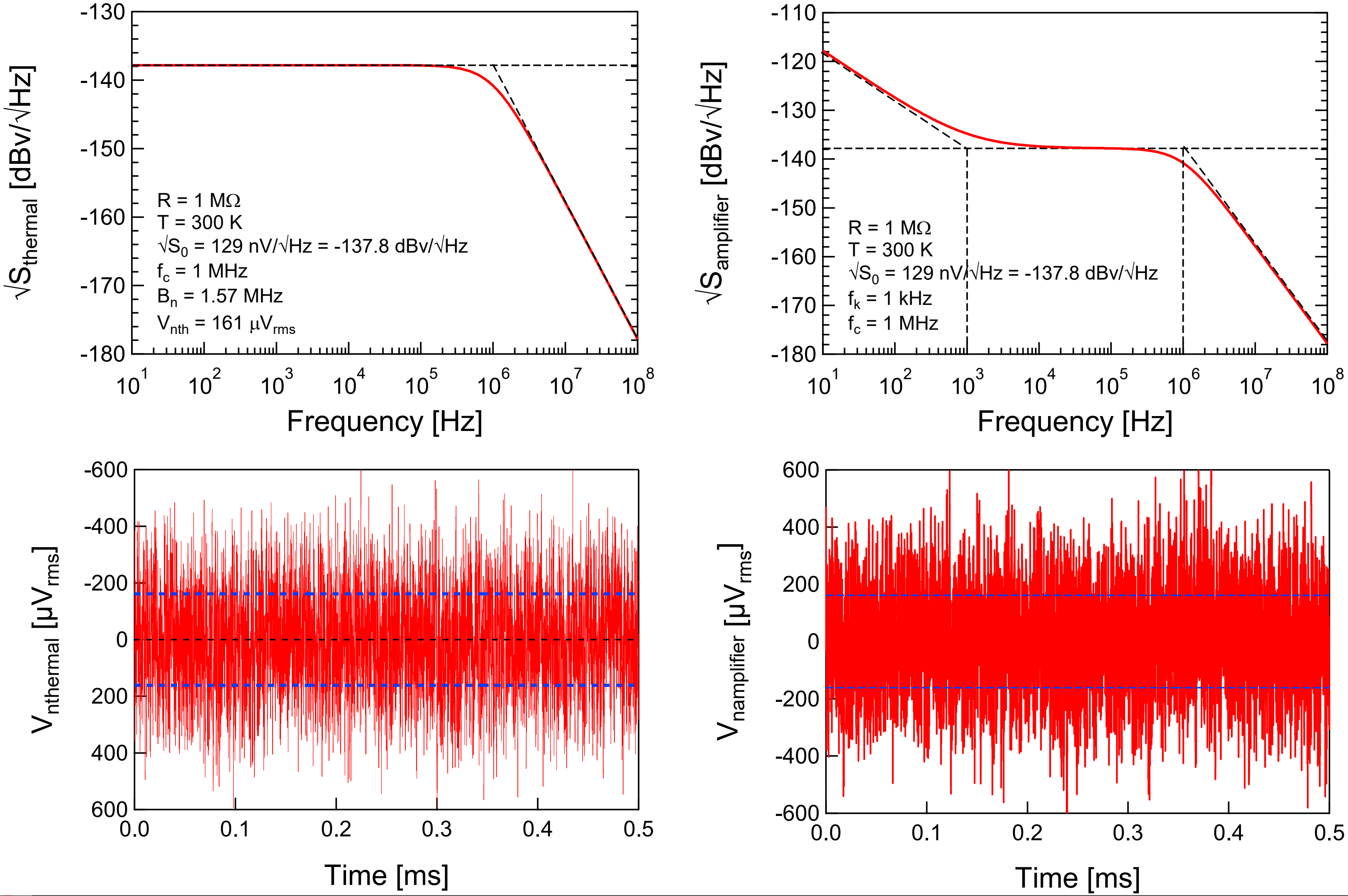
# White Noise


$R_x(\tau)$
$R_x(\tau) = S_0 \cdot \delta(\tau)$
τ
$S_x(f)$
$S_0$
f
$R_x(\tau)$
$2BS_0$
1/2B
τ
$S_x(f)$
$S_0$
f
–B
+B
$R_x(\tau)$
$R_x(\tau) = \pi f_c S_0 e^{-2\pi f_c |\tau|}$
τ
$\frac{-1}{2\pi f_c}$
$\frac{+1}{2\pi f_c}$
$S_x(f)$
$S_0$
$S_0/2$
$S_x(f) = \frac{S_0}{1+(f/f_c)^2}$
f
$-f_c$
$+f_c$

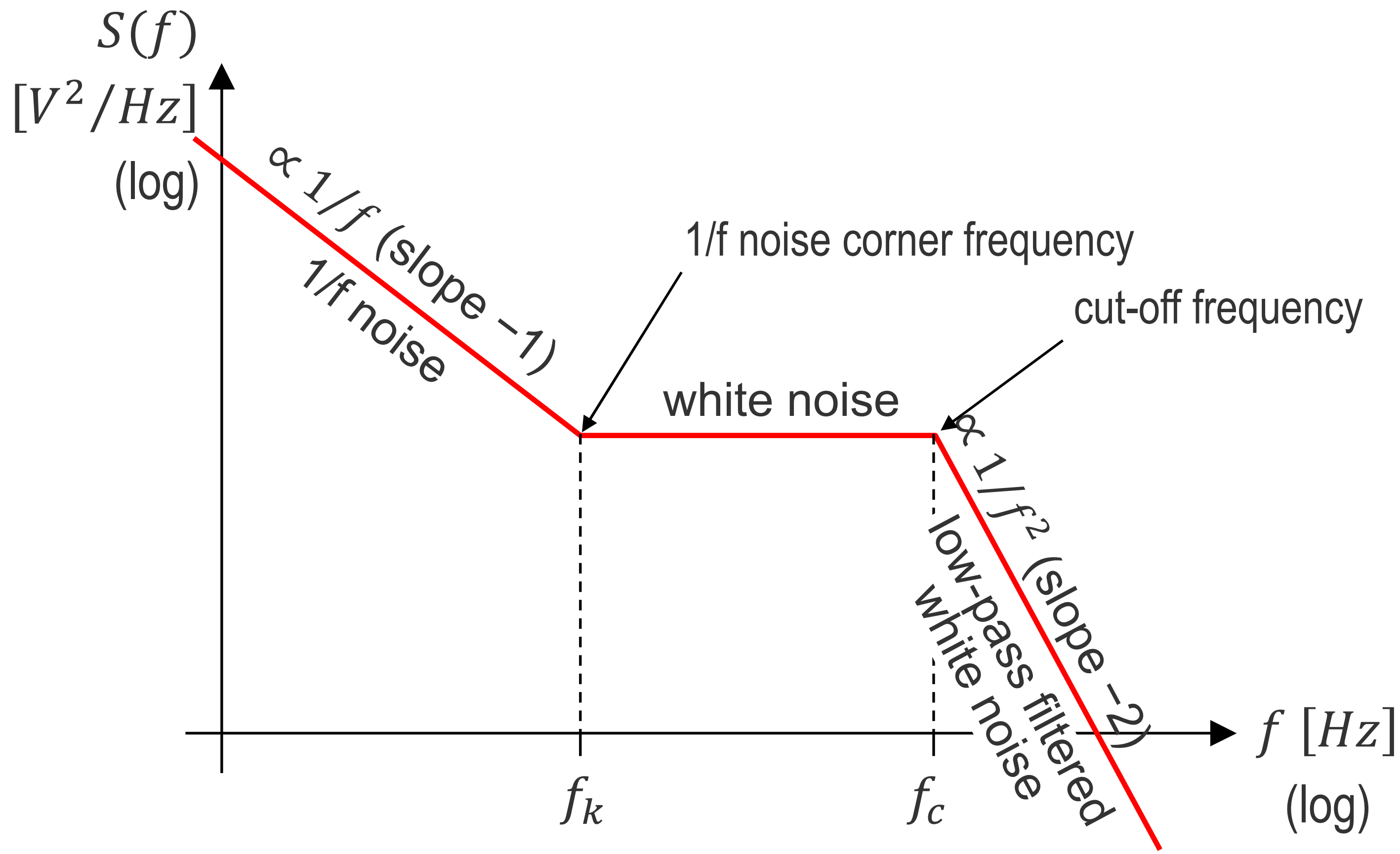
# Noise in Frequency and Time Domain (1/2)



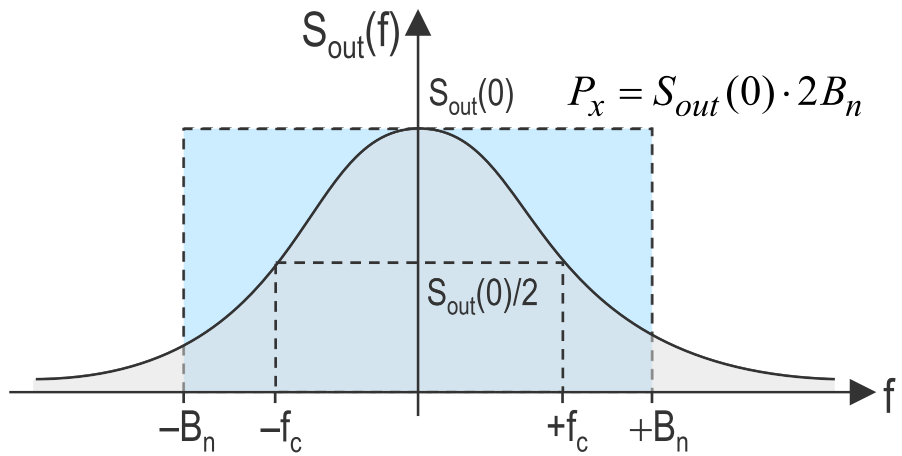
# Noise in Frequency and Time Domain (2/2)



# Typical PSD Shape (output-referred)



# Equivalent Noise Bandwidth



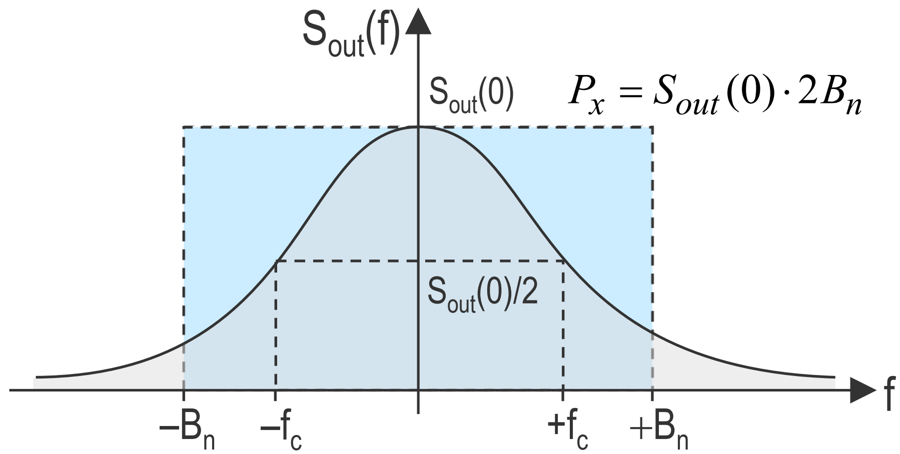

- The **equivalent noise bandwidth** of a low-pass noise PSD is defined as the bandwidth $B_n$ of an ideally low-pass filtered **white noise** having the same value $S_{out}(0)$ (at $f = 0$) and the same power $P_x$ than the original noise

$$B_n = \frac{1}{2} \cdot \frac{1}{S_{out}(0)} \cdot \int_{-\infty}^{+\infty} S_{out}(f) \cdot df = \frac{P_{out}}{2S_{out}(0)}$$

- If the input noise is a white noise of PSD $S_0$, the output noise is then given by $S_{out}(f) = |H(f)|^2 \cdot S_0$ and the **equivalent bandwidth only depends on the transfer function**

$$B_n = \frac{1}{2} \cdot \frac{1}{|H(0)|^2 S_0} \cdot \int_{-\infty}^{+\infty} |H(f)|^2 \cdot S_0 \cdot df = \frac{1}{|H(0)|^2} \cdot \int_{0}^{+\infty} |H(f)|^2 \cdot df$$

# Equivalent Noise Bandwidth of a 1st-order LP Filtered White Noise




- The equivalent noise bandwidth of a **1st-order low-pass filtered white noise** with a cut-off frequency $f_c$ is given by

$$B_n = \int_0^{+\infty} \frac{df}{1+(f/f_c)^2} = f_c \cdot \underbrace{\int_0^{+\infty} \frac{dx}{1+x^2}}_{=\frac{\pi}{2}} = \frac{\pi}{2} f_c$$

# Noise Bandwidth of Various Filters

| Type | Noise Transfer Function | Noise Bandwidth |
|---|---|---|
| 1st-order LP | $\dfrac{1}{1+\dfrac{s}{\omega_c}}$ | $\dfrac{\omega_c}{4}$ |
| 2nd-order LP | $\dfrac{1}{1+\dfrac{s}{\omega_0 \cdot Q}+\left(\dfrac{s}{\omega_0}\right)^2}$ | $\dfrac{\omega_0 \cdot Q}{4}$ |
| 2nd-order BP[1] | $\dfrac{\dfrac{s}{\omega_0 \cdot Q}}{1+\dfrac{s}{\omega_0 \cdot Q}+\left(\dfrac{s}{\omega_0}\right)^2}$ | $\dfrac{\omega_0}{4Q}$ |
| 2nd-order LP (with zero) | $\dfrac{1+\dfrac{s}{\omega_z}}{1+\dfrac{s}{\omega_0 \cdot Q}+\left(\dfrac{s}{\omega_0}\right)^2}$ | $\dfrac{\omega_0 \cdot Q}{4} \cdot \left[1+\left(\dfrac{\omega_0}{\omega_z}\right)^2\right]$ |

1. In the case of the bandpass filter the dc gain $H_0(0)$ is zero and should be replaced in the above definition by the gain at the resonance frequency which in the above case has been set to one.

# Higher Order Transfer Functions

## 3rd-order LP

- Noise transfer function: $$H_{n3}(s) = \frac{n_2 s^2 + n_1 s + n_0}{d_3 s^3 + d_2 s^2 + d_1 s + d_0}$$
- Noise bandwidth: $$B_{n3} = \frac{1}{4} \frac{n_2^2 d_1 d_0 + n_1^2 d_3 d_0 + n_0^2 d_3 d_2 - 2 n_2 n_0 d_3 d_0}{d_3 (d_2 d_1 - d_3 d_0) d_0}$$

## 4th-order LP

- Noise transfer function: $$H_{n4}(s) = \frac{n_3 s^3 + n_2 s^2 + n_1 s + n_0}{d_4 s^4 + d_3 s^3 + d_2 s^2 + d_1 s + d_0}$$
- Noise bandwidth:

$$B_{n4} = \frac{1}{4} \frac{n_3^2 d_3 d_0^2 + 2 n_3 n_1 d_4 d_1 d_0 - n_3^2 d_2 d_1 d_0 - n_2^2 d_4 d_1 d_0 - n_1^2 d_4 d_3 d_0 - n_0^2 d_4 d_3 d_2 + 2 n_2 n_0 d_4 d_3 d_0 + n_0^2 d_4^2 d_1}{d_4 (d_4 d_1^2 - d_3 d_2 d_1 + d_3^2 d_0) d_0}$$

📖 A. Dastgheib and B. Murmann, “Calculation of Total Integrated Noise in Analog Circuits,” TCAS I, vol. 55, no. 10, Nov. 2008.

# Contribution of 1/f Noise to the Total Noise Power

- The PSD (single-sided) of an amplifier including the 1/f noise component can be written as

$$S_n(f) = S_0 \cdot \left(1 + \frac{f_k}{|f|}\right)$$

- where $S_0$ is the white noise component and $f_k$ the corner frequency (frequency at which the 1/f noise becomes equal to the white noise)
- The noise power assuming the noise is filtered by a 1st-order low-pass filter having a cut-off frequency $f_c$ is the given by

$$V_n^2 = \int_0^{+\infty} \frac{S_n(f)}{1 + (f/f_c)^2} \cdot df$$

- We then can define an equivalent noise bandwidth $B_n$ including 1/f noise defined as

$$B_n \triangleq \frac{V_n^2}{S_0} = \int_0^{+\infty} \frac{1 + f_k/f}{1 + (f/f_c)^2} \cdot df = f_c \cdot \int_0^{+\infty} \frac{dx}{1 + x^2} + f_k \cdot \int_{x_\ell}^{+\infty} \frac{dx}{x \cdot (1 + x^2)}$$

- where $x \triangleq f/f_c$ and $x_\ell \triangleq f_\ell/f_c \ll 1$. It is convenient to choose $f_\ell = 1\,Hz$

$$B_n = \frac{\pi}{2} f_c + \frac{f_k}{2} \ln\left[1 + \left(\frac{f_c}{f_\ell}\right)^2\right] \cong \frac{\pi}{2} f_c + f_k \ln\left(\frac{f_c}{f_\ell}\right) = \frac{\pi}{2} f_c + f_k \ln\left(\frac{f_c}{1Hz}\right)$$

- We see that the contribution of 1/f noise to the total noise power is scaling only with the $\ln f_c$ whereas the white noise scales proportionally to $f_c$

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# Main Noise Sources of Circuit Components

Three main sources of noise:

- **Thermal Noise**
  - Due to thermal excitation of charge carriers
  - Appears as white spectral density
- **Shot Noise**
  - Due to carriers randomly crossing a barrier
  - Depends on dc bias current and is white
- **Flicker Noise**
  - Due to traps in semiconductors
  - Has a 1/f spectral density
  - Significant in MOS transistors at low frequencies

# Thermal Noise – The Equipartition Theorem

- Every closed physical system at temperature $T$ contains energy of average amount $kT/2$ per degree of freedom, where $k = 1.38 \cdot 10^{-23}\, J/K$ is the Boltzmann constant
- Consider a gas of electrons having a Maxwellian velocity distribution

$$p(v) = \sqrt{\frac{m}{2\pi kT}} \cdot e^{-\frac{mv^2}{2kT}}$$

  where $p(v) \cdot dv$ represents the probability of finding one electron having a velocity comprised between $v$ and $v + dv$
- The average energy of the electron at equilibrium is given by

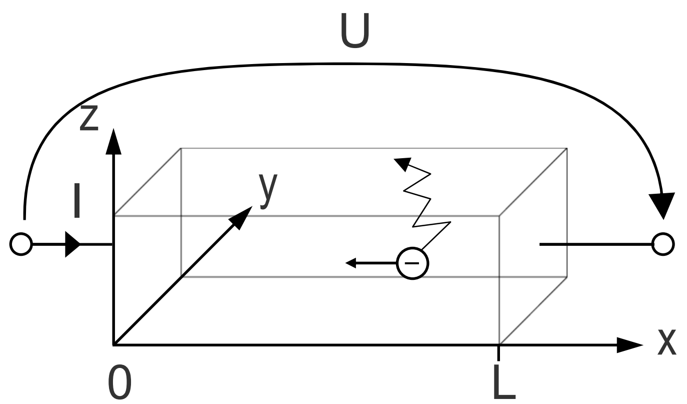
$$\overline{W} = E\left[m\frac{v^2}{2}\right] = \frac{m}{2}\, E[v^2] = \frac{m}{2} \cdot \int_0^{+\infty} v^2 \cdot p(v) \cdot dv = \frac{kT}{2}$$

- Hence, for a one degree of freedom system, we have

$$\frac{m\overline{v^2}}{2} = \frac{kT}{2}$$

📖 C. Kittel, *Elementary Statistical Physics*, John Wiley, 1958.

# The Nyquist Theorem – Microscopic Derivation




$A$: section

$n$: electron density

- The voltage is given by Ohm's law

$$U = R \cdot I = R \cdot A \cdot J = R \cdot A \cdot n \cdot q \cdot v$$

- where $A$ is the section, $n$ the electron density and $v$ is the **drift velocity** along the $x$ axis, averaged over the ensemble of electrons $N$

$$v = \frac{1}{N} \cdot \sum_i v_i \text{ with } N = n \cdot A \cdot L$$

- The voltage $U$ is then given by

$$U = \frac{q \cdot R}{L} \cdot \sum_i v_i = \sum_i u_i \text{ where } u_i = \frac{q \cdot R}{L} \cdot v_i$$

C. Kittel, *Elementary Statistical Physics*, John Wiley, 1958.

# The Nyquist Theorem – Microscopic Derivation

- If no dc current flows, then $I = 0$, $U = 0$ and $v = 0$
- On the other hand the variances are non-zero

$$\overline{v^2} = E[v^2] \neq 0 \text{ and } \overline{U^2} = E[U^2] \neq 0$$

- Assuming no correlation between electrons, the ACF of $U$ is given by

$$R_u(\tau) = \sum_i R_{u_i}(\tau) = \left(\frac{q \cdot R}{L}\right)^2 \cdot \sum_i R_{v_i}(\tau) \text{ where } R_{v_i}(\tau) \text{ is the ACF of } v_i$$

- If $v_i$ has a Maxwellian distribution then $R_{v_i}(\tau)$ is given by

$$R_{v_i}(\tau) = R_{v_i}(0) \cdot e^{-\frac{|\tau|}{\tau_0}}$$

- where $\tau_0$ is the **relaxation time** or mean time of flight of the conduction electrons
- $R_{v_i}(0)$ can be found using the equipartition theorem according to

$$m\frac{v_i^2}{2} = \frac{m}{2} R_{v_i}(0) = \frac{kT}{2} \text{ which results in } R_{v_i}(0) = \frac{kT}{m}$$

- The ACF of the voltage is then simply given by

$$R_u(\tau) = N \cdot \left(\frac{q \cdot R}{L}\right)^2 \cdot \frac{kT}{m} \cdot e^{-\frac{|\tau|}{\tau_0}}$$

C. Kittel, *Elementary Statistical Physics*, John Wiley, 1958.

# The Nyquist Theorem – Microscopic Derivation

- The corresponding bilateral PSD of the noise voltage is a **Lorentzian spectrum**

$$S_u(f) = N \cdot \left(\frac{q \cdot R}{L}\right)^2 \cdot \frac{kT}{m} \cdot \frac{2\tau_0}{1 + (2\pi f \tau_0)^2}$$

- Usually in metals at room temperature $\tau_0 < 10^{-13} s$ and hence $2\pi f \tau_0 \ll 1$

$$S_u(f) \cong N \cdot \left(\frac{q \cdot R}{L}\right)^2 \cdot \frac{kT}{m} \cdot 2\tau_0 = 2kTR \cdot \frac{n \cdot A \cdot q^2 \cdot R \cdot \tau_0}{m \cdot L} \text{ for } f \ll \frac{1}{2\pi\tau_0}$$

- Recalling that the conductivity $\sigma$ is given by $\sigma = q\mu n$ and the mobility $\mu$ by $\mu = q\tau_0/m$, the resistance can then be written as

$$R = \frac{L}{\sigma \cdot A} = \frac{L}{A} \cdot \frac{m}{q^2 \cdot n \cdot \tau_0}$$

- Replacing $R$ in the above right term leads to the **bilateral** PSD of thermal noise

$$S_u(f) = 2kTR$$

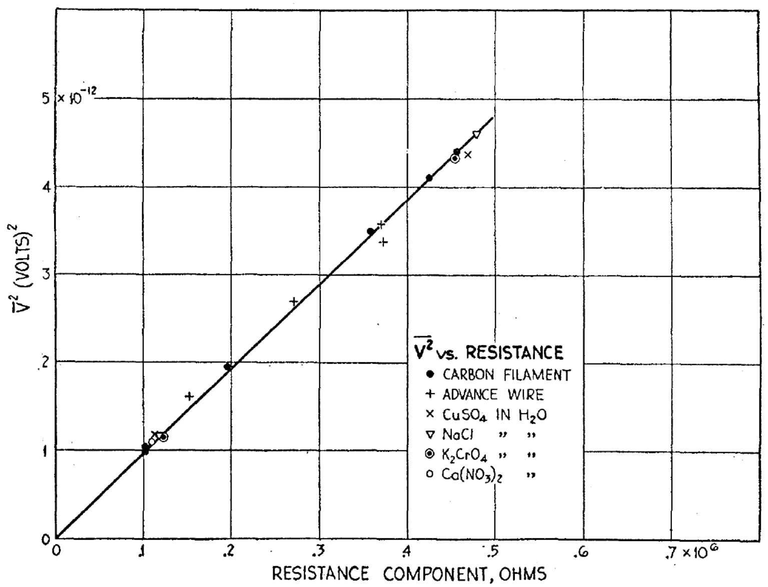
- or in **unilateral** form

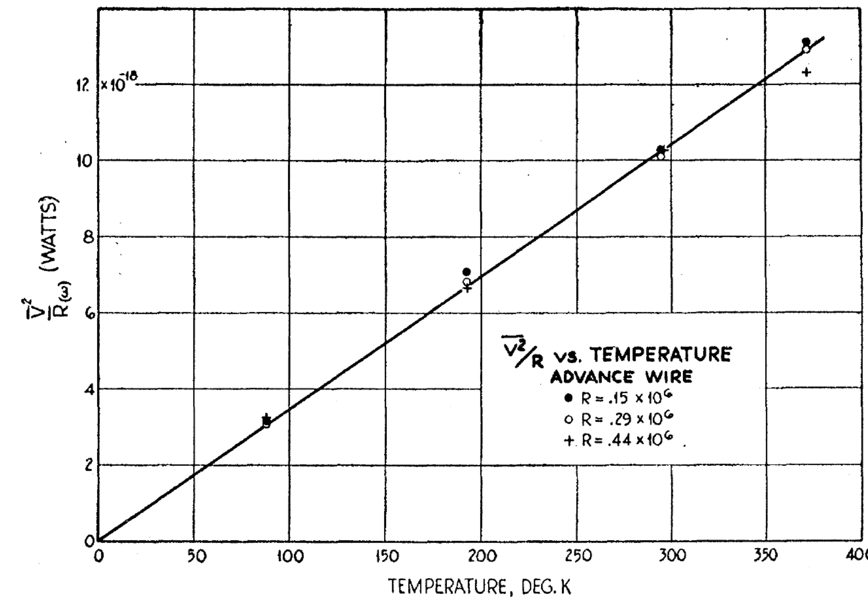
$$S_u(f) = 4kTR$$

C. Kittel, *Elementary Statistical Physics*, John Wiley, 1958.

# Thermal Noise – The Original Measurements by Johnson

$$S_{th} = 4kT \cdot R$$

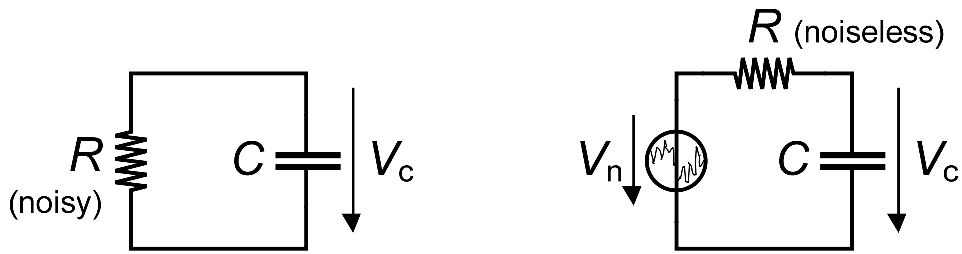



- The thermal noise PSD is
  - proportional to the resistance $R$ (left figure)
  - proportional to temperature $T$ (right figure)
  - independent of the current flowing through the resistor

J. B. Johnson, "Thermal Agitation of Electricity in Conductors," Physical Review, vol. 32, pp. 97-109, July 1928.

# Thermal Noise – $kT/C$ Noise




- The **variance** of the thermal noise voltage $V_c$ across $C$ is given by

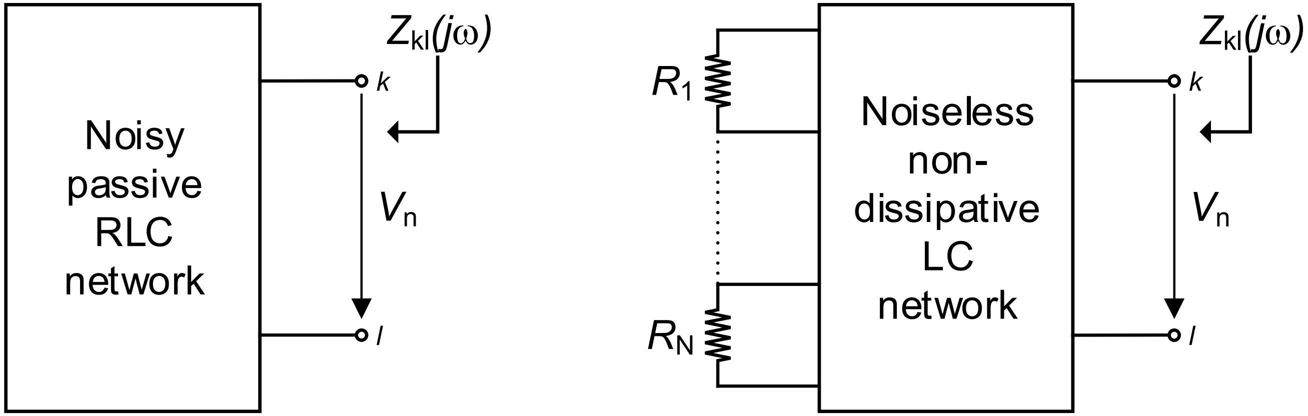
$$\overline{V_c^2} = 4kTR \cdot \int_0^{+\infty} \frac{df}{1+(2\pi f\tau)^2} = 4kTR \cdot B_n = 4kTR \cdot \frac{\pi}{2}\frac{1}{2\pi RC} = \frac{kT}{C}$$

- This result can be obtained directly by applying the **equipartition theorem**
- Average stored energy in $C$ is given by $\overline{W} = \frac{1}{2} C \cdot \overline{V_c^2}$
- Since resistor $R$ and capacitor $C$ are in thermal equilibrium and there is only one degree of freedom, we have

$$\overline{W} = kT/2$$

- We then get $$C \cdot \overline{V_c^2} = kT \Longrightarrow \overline{V_c^2} = \frac{kT}{C}$$

# Thermal Noise in Passive Networks – The Nyquist Theorem




- The power spectral density (PSD) of noise voltage $V_n$ is given by

$$S_{v_n}(f) = 4kT \cdot \Re\{Z_{kl}(j2\pi f)\}$$

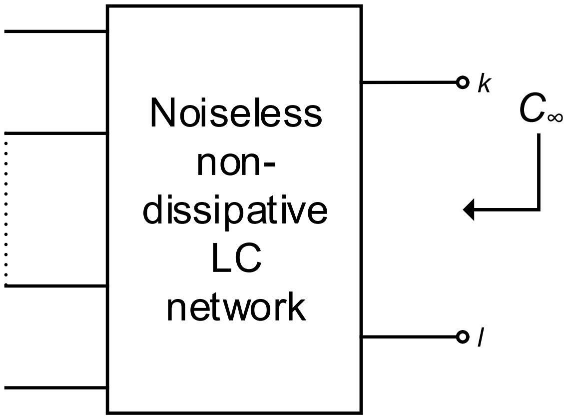
- The variance of voltage $V_n$ is then given by

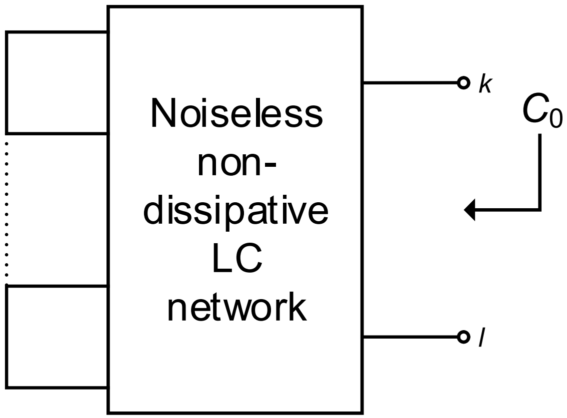
$$\overline{V_n^2} = \int_0^{+\infty} S_{v_n}(f) \cdot df = 4kT \cdot \int_0^{+\infty} \Re\{Z_{kl}(j2\pi f)\} \cdot df$$

- Or we can use the Bode theorem given in the next slide


📖 H. Nyquist, "Thermal Agitation of Electric Charge in Conductors," Physical Review B, vol. 32, pp. 110-113, July 1928.
📖 A. Papoulis, *Probability, Random Variables and Stochastic Processes*, 1st ed., pp. 362-363, 1981.

# Thermal Noise in Passive Networks – The Bode Theorem




$$\frac{1}{C_\infty} = \lim_{s \to \infty} [s \cdot Z(s)] \qquad \frac{1}{C_0} = \lim_{s \to 0} [s \cdot Z(s)]$$

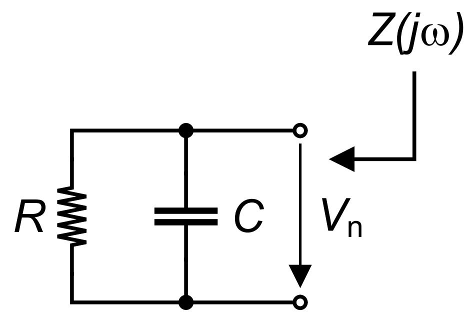
- The variance of noise voltage $V_n$ can be obtained without computing the integral by using the Bode theorem stating

$$\overline{V_n^2} = kT \cdot \left[\frac{1}{C_\infty} - \frac{1}{C_0}\right]$$

- Where $C_\infty$ and $C_0$ are define as follows

H. W. Bode, *Network Analysis and Feedback Amplifier Design*, van Nostrand Company, New York, pp. 281-282, 1945.

# Nyquist and Bode Theorems – Example (1/2)




- The impedance is given by

$$Z(j\omega) = \frac{1}{1/R + j\omega C} = \frac{R}{1 + j\omega RC} = \frac{R}{1 + (\omega RC)^2} - j\frac{\omega R^2 C}{1 + (\omega RC)^2}$$

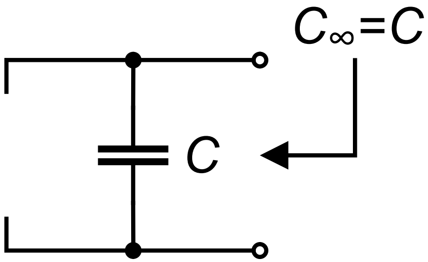
- And hence $S_{v_n}(f) = 4kT \cdot \Re\{Z(j2\pi f)\} = \frac{4kTR}{1+(\omega RC)^2}$

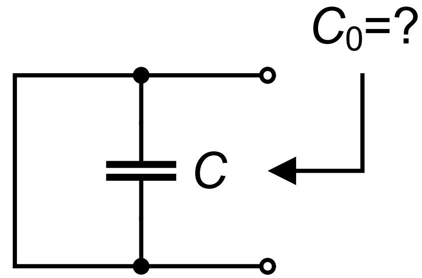
- The noise variance is then given by

$$\overline{V_n^2} = \int_0^{+\infty} S_{v_n}(f) \cdot df = 4kTR \cdot \int_0^{+\infty} \frac{df}{1 + (2\pi f \tau)^2} = 4kTR \cdot B_n$$

$$= 4kTR \cdot \frac{\pi}{2} \frac{1}{2\pi RC} = \frac{kT}{C}$$

# Nyquist and Bode Theorems – Example (2/2)




$$\frac{1}{C_\infty} = \lim_{s \to \infty} [s \cdot Z(s)] = \lim_{s \to \infty} \frac{sR}{1 + sRC} = \frac{1}{C} \qquad \frac{1}{C_0} = \lim_{s \to 0} [s \cdot Z(s)] = \lim_{s \to 0} \frac{sR}{1 + sRC} = 0$$

- Noise variance of a 1st-order RC circuit using Bode theorem results in

$$\overline{V_n^2} = kT \cdot \left[\frac{1}{C_\infty} - \frac{1}{C_0}\right] = kT \cdot \left[\frac{1}{C} - 0\right] = \frac{kT}{C}$$

- Just by circuit inspection without any integration!
- However, unfortunately only applies to passive circuits!

# Shot Noise – The Poisson Process

- The Poisson process is characterized by a **sequence of independent random events**, occurring at any time $t_k$ with the **same probability**
- The probability to have exactly $n$ events in the time interval $[0, t]$ is given by

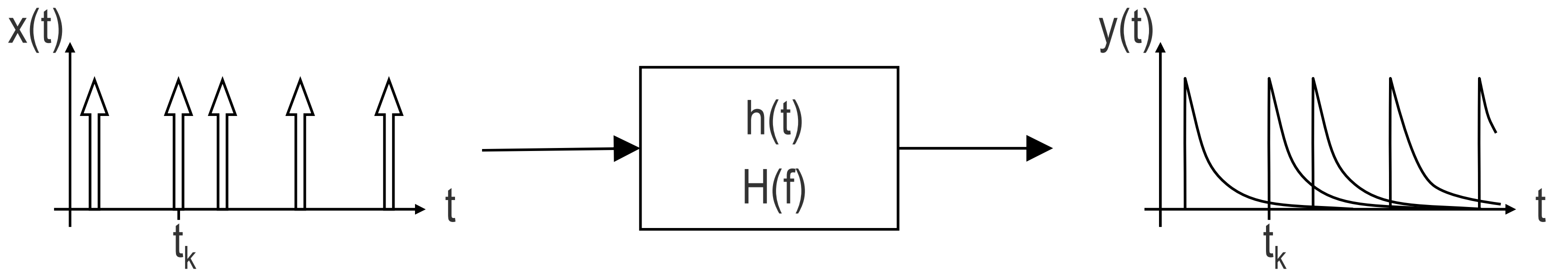
$$p_n(t) = \frac{(\lambda \cdot t)^n}{n!} \cdot e^{-\lambda \cdot t}$$

- where $\lambda$ is the **average number of events per second**, which can be assumed to be constant
- The average (or expected) number of events therefore grows **linearly with time**

$$E[n] = \sum_{n=0}^{+\infty} n \cdot p_n(t) = \lambda \cdot t$$

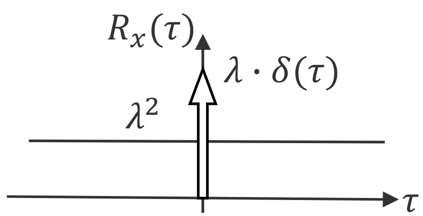
A. Papoulis, *Probability, Random Variables and Stochastic Processes*, 1st ed., 1981.

# Shot Noise – The Shot Noise Process

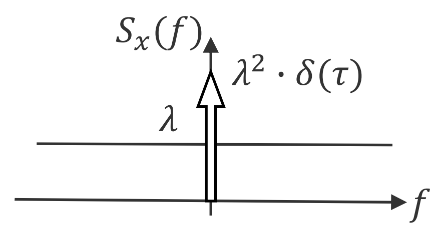



$$x(t) = \sum_k \delta(t - t_k) \qquad y(t) = \sum_k h(t - t_k)$$

- The **shot noise** process is defined by $y(t) = \sum_k h(t - t_k)$, where $t_k$ are random points in time with uniform density $\lambda$
- $y(t)$ can be considered as the output of a linear system having an impulse response $h(t)$ and a sequence of **Poisson impulses** at the input $x(t) = \sum_k \delta(t - t_k)$
- It can be shown that the PSD is made of a **DC component $\lambda^2 \cdot \delta(f)$** and an additional **white noise $\lambda$**

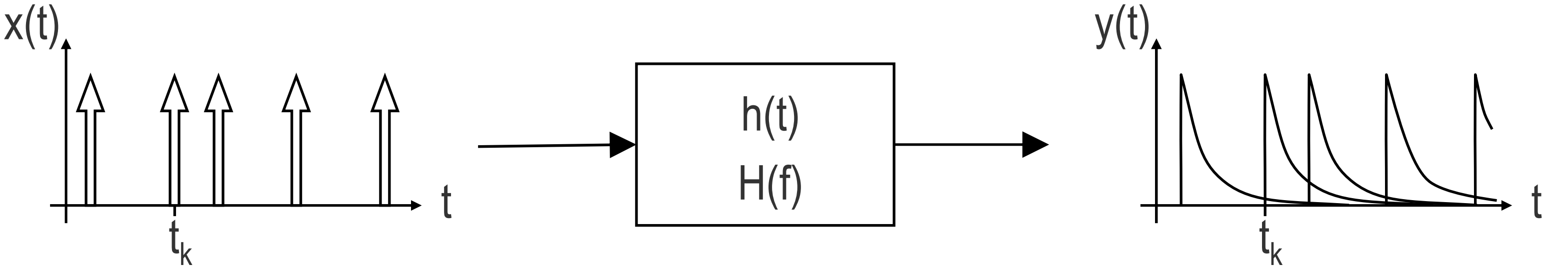



$$R_x(\tau) = \lambda^2 + \lambda \cdot \delta(\tau)$$
$$S_x(f) = \lambda^2 \cdot \delta(\tau) + \lambda$$

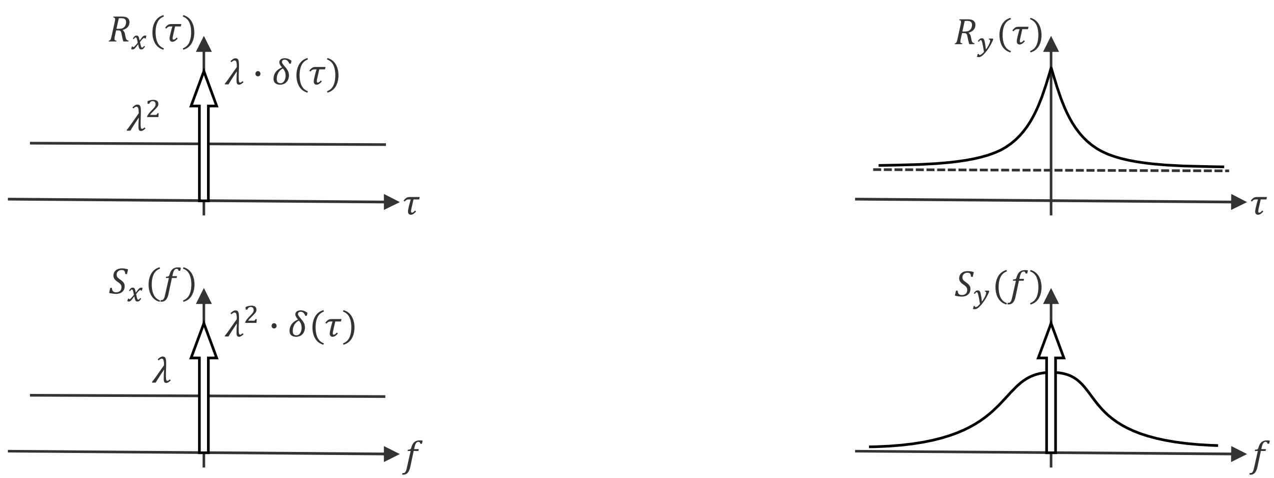



A. Papoulis, *Probability, Random Variables and Stochastic Processes*, 1st ed., 1981.

# Shot Noise – The Shot Noise Process




$$x(t) = \sum_k \delta(t - t_k) \qquad y(t) = \sum_k h(t - t_k)$$

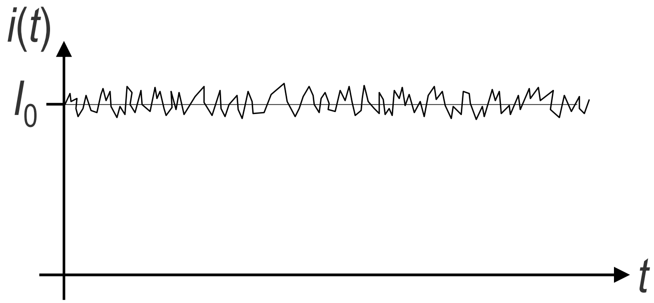


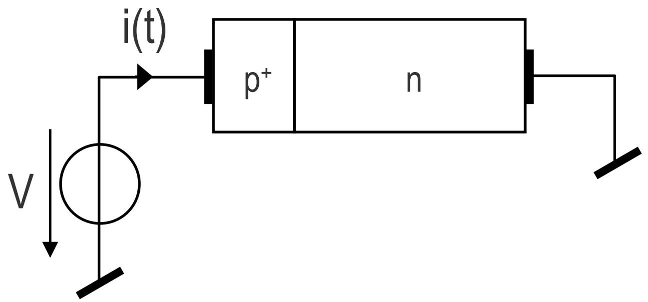

$$R_x(\tau) = \lambda^2 + \lambda \cdot \delta(\tau)$$
$$S_x(f) = \lambda^2 \cdot \delta(\tau) + \lambda$$

$$R_y(\tau) = \lambda^2 \cdot |H(0)|^2 + \lambda \cdot \int_{-\infty}^{+\infty} h(t+\tau) \cdot h(t) \cdot dt$$
$$S_y(f) = \lambda^2 \cdot |H(0)|^2 \cdot \delta(\tau) + \lambda \cdot |H(f)|^2$$

📖 A. Papoulis, *Probability, Random Variables and Stochastic Processes*, 1st ed., 1981.

# Shot Noise – Current Through a p-n Junction



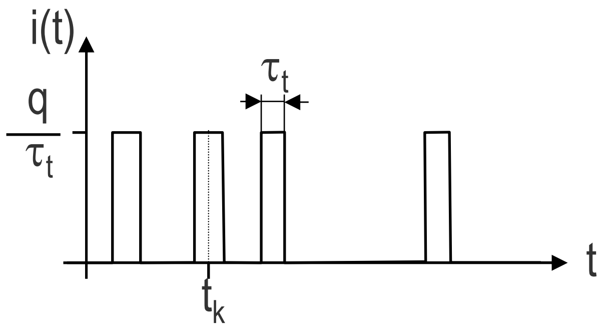

- Neglecting the generation-recombination in the space charge zone

$$I_d = I_s \cdot \left(e^{\frac{V}{U_T}} - 1\right) \text{ with } U_T \triangleq \frac{kT}{q}$$

- Current $I_d$ is composed of:
  - $I_d + I_s = I_s \cdot e^{V/U_T}$: holes injected into the n region recombining there or reaching the ohmic contact
  - $-I_s$: holes generated in the n region and collected by the p region

A. van der Ziel, *Noise in Solid-State Devices and Circuits,* Wiley, 1986.

# Shot Noise – Current Through a p-n Junction (cont.)




$\tau_t$: average transit time

$$h(t) = \frac{q}{\tau_t} \cdot rect\left(\frac{t}{\tau_t}\right)$$

with $rect(x) = \begin{cases} 1 & |x| < \frac{1}{2} \\ 0 & otherwise \end{cases}$

- Assuming that each carrier has the same probability to cross the barrier at any time and that the average number of carrier crossing the barrier per unit time remains constant and equal to $\lambda$, the current $i(t)$ is a **shot noise process** with

$$\lambda \cdot q = (I_d + I_s) + I_s = I_d + 2I_s = I_{eq}$$

$$S_I(f) = (\lambda \cdot q)^2 \cdot \delta(f) + \lambda \cdot q^2 \cdot sinc^2(\pi f \tau_t) = I_{eq}^2 \cdot \delta(f) + q \cdot I_{eq} \cdot sinc^2(\pi f \tau_t)$$

- The bilateral PSD of the current fluctuation is then given by

$$S_{\Delta I} = q \cdot I_{eq} \cdot sinc^2(\pi f \tau_t) \cong q \cdot I_{eq} = q \cdot (I_d + 2I_s) \text{ for } f \ll 1/(\pi \tau_t)$$

- Or in unilateral PSD the well-known expression

$$S_{\Delta I} = 2q \cdot I_{eq} \cong 2q \cdot I_d$$

A. van der Ziel, *Noise in Solid-State Devices and Circuits,* Wiley, 1986.

# Shot Noise – Current Through a p-n Junction (cont.)

- The unilateral PSD can also be written in terms of the small-signal differential conductance $G_d$

$$G_d = \frac{dI_d}{dV} = \frac{I_s}{U_T} \cdot e^{\frac{V}{U_T}} = \frac{I_d + I_s}{U_T}$$

- as it would be thermal noise

$$S_{\Delta I} = 2kT \cdot G_d \cdot \frac{I_d + 2I_s}{I_d + I_s}$$

- corresponding to full thermal noise of the conductance $G_{d0} = I_s/U_T$ at zero bias ($V = 0$ and $I_d = 0$)

$$S_{\Delta I} = 4kT \cdot G_{d0}$$

- and "half thermal noise" for $I_d \gg I_s$

$$S_{\Delta I} = 2kT \cdot G_d = 2kT \cdot \frac{I_d}{U_T} = 2qI_d$$

A. van der Ziel, *Noise in Solid-State Devices and Circuits,* Wiley, 1986.

# Flicker or 1/f Noise (1/2)

- Random process with zero mean and characterized by its PSD

$$S_{1/f}(f) = \frac{K}{|f|^{\alpha}}$$

- with $\alpha$ close to one and where $K$ is a constant
- The power (variance) in bandwidth $[f_\ell, f_h]$ assuming $\alpha = 1$ is given by

$$\sigma_{1/f}^2 = 2 \cdot \int_{f_\ell}^{f_h} \frac{K}{f} \cdot df = 2K \cdot \ln\left(\frac{f_h}{f_\ell}\right)$$

- The variance tends to infinity as $f_h \to \infty$ or $f_\ell \to 0$
- Divergence of variance at high frequency does not cause a problem since it is always low-pass filtered
- On the other hand, divergence of variance when $f_\ell \to 0$ causes many controversies and interrogations!

M. S. Keshner, "1/f noise," Proc. of the IEEE, vol. 70, pp. 212-218, 1982.

# Flicker or 1/f Noise (2/2)

- The "problem" of the divergence of the variance when $f_\ell \rightarrow 0$ can be "solved" when considering the process as **non-stationary**

$$R_{1/f}(t_2, \tau) \approx f(t_2) + f(\tau) \text{ for } 0 < \tau \ll t_2$$

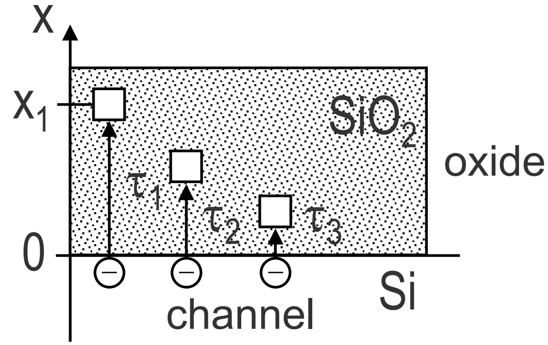
where $t_2$ is the **age** of the process and $\tau = t_2 - t_1$

- The corresponding PSD follows a 1/f law down to frequency corresponding to the observation time $T_{obs}$ which is independent of the values of $t_2$ and $T_{obs}$
- 1/f noise has been observed as fluctuations not only in semiconductors and electronic devices but also in many very different systems, including:
  - average seasonal temperature,
  - annual amount of rainfall,
  - rate of traffic flow,
  - economic data,
  - the loudness and pitch of music, etc…

M. S. Keshner, "1/f noise," Proc. of the IEEE, vol. 70, pp. 212-218, 1982.

# Flicker Noise in the MOST – The Mc-Worther Model (1/3)

cross-section of a MOS transistor showing the mobile carriers (electrons) trapped into the oxide




tunneling time constant

$$\tau_t = \tau_0 \cdot e^{\alpha \cdot x}$$

where $\alpha \approx 10^8\ cm^{-1}$

- Carrier density fluctuation due to **trapping** via tunneling effect in traps located in the oxide and close to the Si-$SiO_2$ interface
- Tunneling effect characterized by **tunneling time constant $\tau_t$**
- The PSD corresponding to the fluctuation $\Delta N$ in number of electrons (holes) $N$ due to a single trap having a tunneling time constant $\tau_t$ is given by a **Lorentzian PSD**

$$S_{\Delta N}(f) = \overline{\Delta N^2} \cdot \frac{2\tau_t}{1 + (2\pi f \tau_t)^2}$$

- where $\overline{\Delta N^2}$ is the variance of $\Delta N$

📖 A. van der Ziel, *Noise in Solid-State Devices and Circuits,* Wiley, 1986.

# Flicker Noise in the MOST – The Mc-Worther Model (2/3)

- The variance $\overline{\Delta N^2}$ of $\Delta N$ can be assumed to be proportional to $N$ and hence $\overline{\Delta N^2} = \beta \cdot N$ resulting in

$$S_{\Delta N}(f) = \beta \cdot N \cdot \frac{2\tau_t}{1 + (2\pi f \tau_t)^2}$$

- It can be shown that the PSD of the drain current resulting from the charge fluctuation in the channel due to a single trap is given by

$$S_{\Delta I_D}(f) = \left(\frac{I_D}{N}\right)^2 \cdot S_{\Delta N}(f) \cong \frac{\beta \cdot I_D^2}{N} \cdot \frac{2\tau_t}{1 + (2\pi f \tau_t)^2}$$

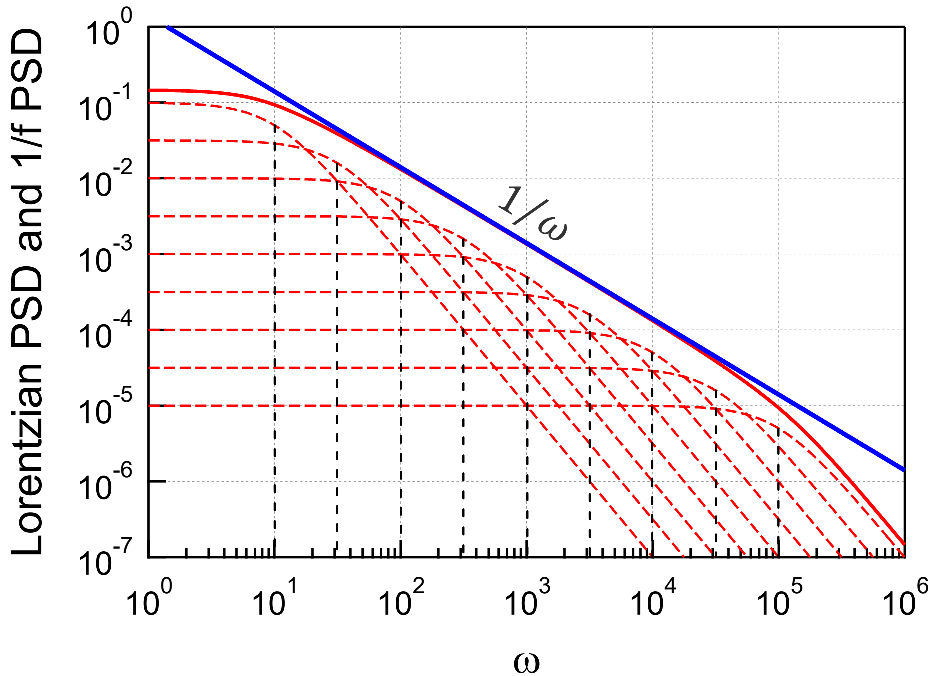
- Averaging over all time constant $\tau_{tmin} \leq \tau_t \leq \tau_{tmax}$ results in a 1/f PSD within $1/(2\pi\tau_{tmax})$ and $1/(2\pi\tau_{tmin})$

$$\frac{S_{\Delta I_D}(f)}{I_D^2} = \frac{K_f}{f}$$

- with $$K_f = \frac{\beta}{\pi N} \cdot \frac{atan(2\pi f \tau_{t,max}) - atan(2\pi f \tau_{t,min})}{\ln(\tau_{t,max}/\tau_{t,min})}$$

A. van der Ziel, *Noise in Solid-State Devices and Circuits,* Wiley, 1986.

# Flicker Noise in the MOST – The Mc-Worther Model (3/3)




- The above integration corresponds to summing all the Lorentzian curves corresponding to each trap results in a 1/f power spectrum
- This is illustrated in the above plot with 9 Lorentzians equally spaced (in log scale)

$$S(\omega) = \frac{\tau_t}{1 + (\omega \tau_t)^2} = \frac{1}{\omega_t} \cdot \frac{1}{1 + \left(\frac{\omega}{\omega_t}\right)^2}$$

# Flicker Noise in the MOST – The Hooge Model

- Noise due to fluctuations of the **carrier mobility**
- **Volume** effect rather than **interface** effect like in the Mc-Worther model
- The current fluctuation PSD in semiconductor can be expressed as

$$S_{\Delta I}(f) = \frac{\alpha_H}{f \cdot N} \cdot I^2$$

- where $\alpha_H \cong 2 \times 10^{-3}$ is the Hooge parameter, $I$ is the bias current and $N$ the total number of mobile charge
- Applied to the MOS transistor, the flicker noise is often referred to the gate (fluctuations of the gate voltage instead of fluctuations of the drain current)
- In strong inversion and saturation (with $V_S = 0$), this model leads to a PSD

$$S_{\Delta V_G}(f) \cong \frac{\alpha_H \cdot q \cdot (V_G - V_{T0})}{2W \cdot L \cdot C_{ox} \cdot f}$$

- Inversely proportional to gate area $W \cdot L$ and to $C_{ox}$ and proportional to $V_G - V_{T0}$

📖 F. M. Klaassen, “Characterization of Low 1/f Noise in MOS Transistors,” IEEE Transactions on Electron Devices, vol. 18, no. 10, pp. 887-891, Oct. 1971.
📖 A. van der Ziel, *Noise in Solid-State Devices and Circuits,* Wiley, 1986.

# Random Telegraph Noise (RTN)

- CMOS technology scaling led to reduced gate area and lower current levels uncovering the random telegraph noise (RTN)
- Due to trapping and detrapping of mobile charge characterized by very long capturing and emitting time constants $\tau_c$ and $\tau_e$
- The PSD of the induced drain current fluctuation due to a single trap is given by

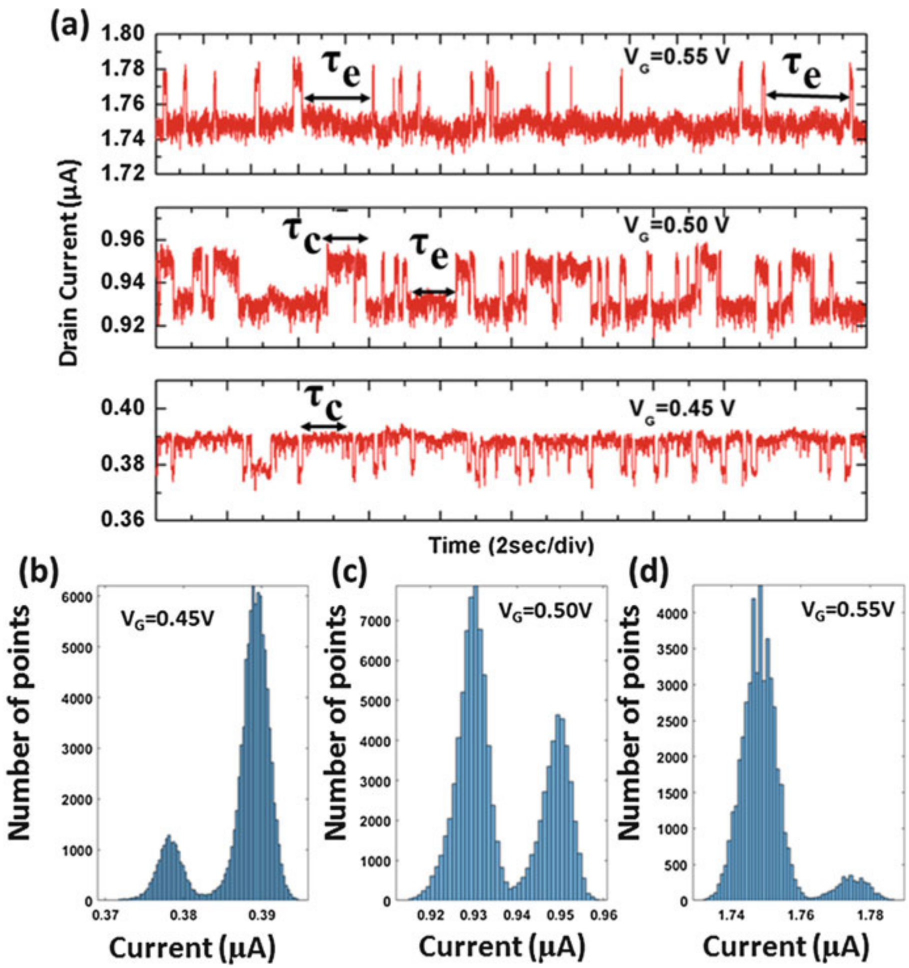
$$\frac{S_{\Delta I_D^2}}{I_D^2} = \frac{\overline{\Delta N^2}}{N^2} \cdot \frac{4\tau_r}{1 + (2\pi f \tau_r)^2}$$

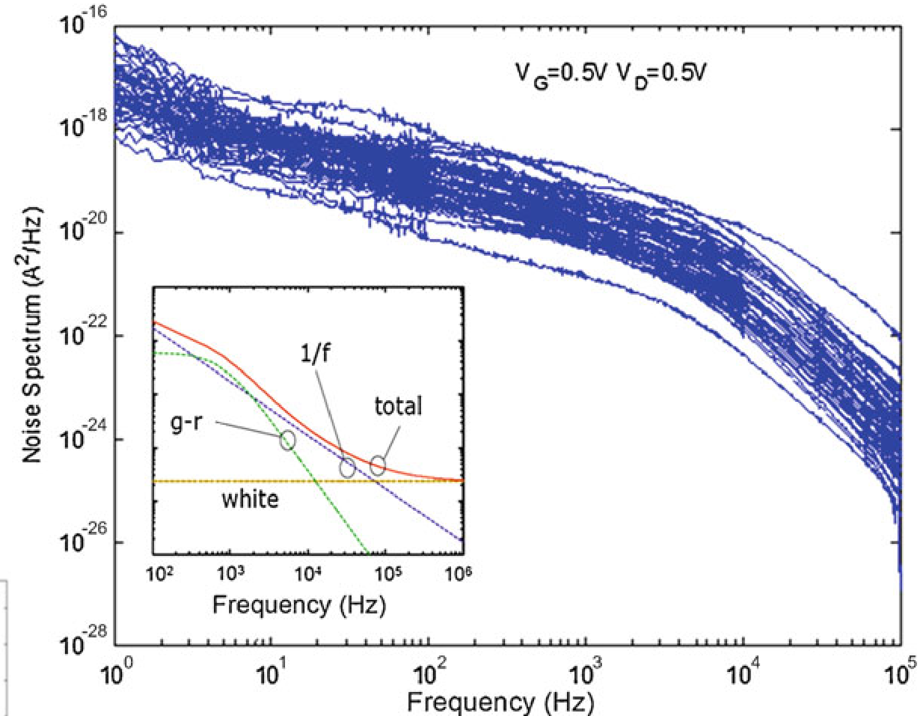
- where $\tau_r = (1/\tau_c + 1/\tau_e)^{-1}$ is the carrier lifetime or effective time constant, $\overline{\Delta N^2}$ is the variance of the fluctuations of the number of carriers in the channel and $N$ is the average number of carriers in the channel
- Note that $\overline{\Delta N^2}$ is proportional to the concentration of generation–recombination-trapping centers
- The PSD of the current fluctuations due to a single trap is then a Lorentzian spectrum given by

$$S_{\Delta I_D^2} = 4\, I_D^2 \, \frac{\tau_r}{\tau_e + \tau_c} \frac{\tau_r}{1 + (2\pi f \tau_r)^2}$$

C. Marquez, *et al.*, “Systematic Characterization of Random Telegraph Noise and Its Dependence with Magnetic Fields in MOSFET Devices,” in *Noise in Nanoscale Semiconductor Devices*, ed. T. Grasser (ed.), Springer 2020. https://doi.org/10.1007/978-3-030-37500-3_4.

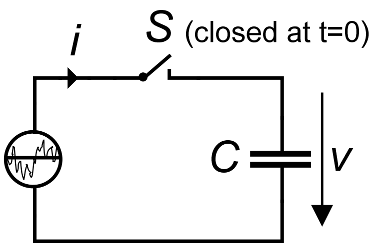
# RTN Time Signature and PSD




C. Marquez, *et al.*, “Systematic Characterization of Random Telegraph Noise and Its Dependence with Magnetic Fields in MOSFET Devices,” in *Noise in Nanoscale Semiconductor Devices*, ed. T. Grasser (ed.), Springer 2020. https://doi.org/10.1007/978-3-030-37500-3_4.

# Integrated White Noise – Random Walk (Wiener Process) (1/2)




- The integration of a stationary white (current) noise having a PSD and ACF given by

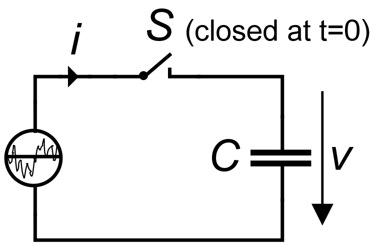
$$S_i(f) = S_0 \text{ and } R_i(\tau) = S_0 \cdot \delta(\tau)$$

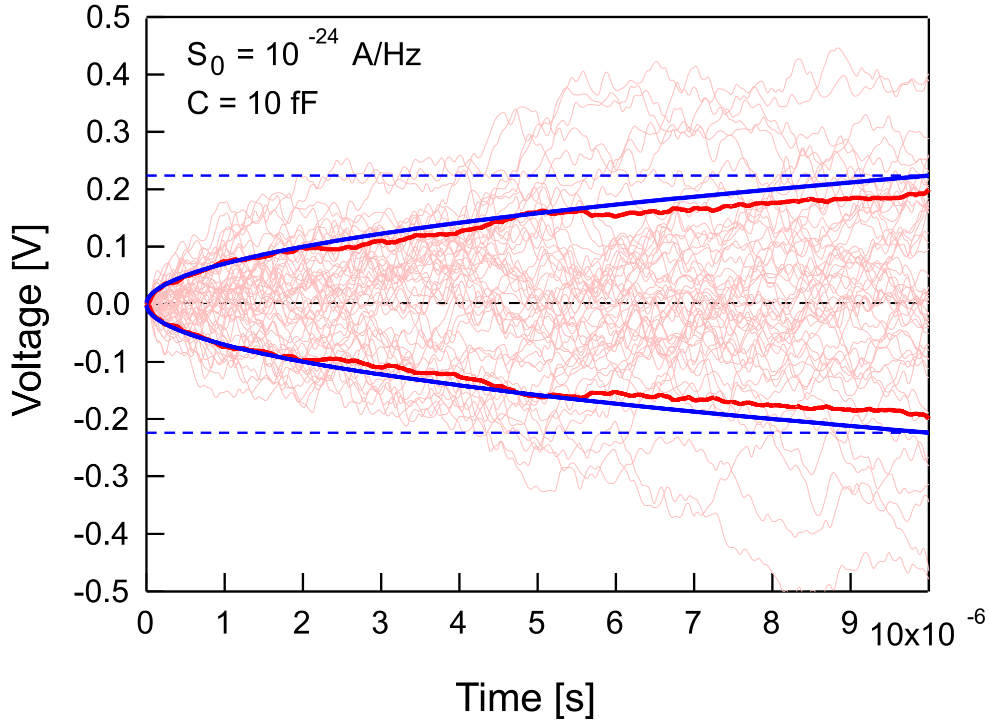
- Gives rise to a non-stationary (voltage) noise with an ACF given by

$$R_{vv}(t,\tau) = \begin{cases} \frac{S_0}{C^2} \cdot t & 0 < t < T \\ \frac{S_0}{C^2} \cdot T & 0 < T < t \end{cases}$$

- Note that $S_0$ is the bilateral FT i.e. if $i(t)$ is thermal noise then $S_0 = 2kT \cdot R$

# Integrated White Noise – Random Walk (Wiener Process) (2/2)




- The standard deviation of the voltage (rms value) is therefore increasing with $\sqrt{t}$
- The above plot shows the standard deviation obtained from transient noise simulations

# Other Types of Noise

- Generation-recombination noise
  - Fluctuation of the conductance due to fluctuating occupancy of the generation-recombination centers
- Quantization noise
  - Noise introduced by the quantization of analog signal
  - Not really a random signal, but can be considered as if signal is “busy” and there are many quantization steps
  - If the quantization error is uniformly distributed over the quantization step, then the variance is given by

$$\sigma_q^2 \approx \frac{\Delta^2}{12}$$ where $\Delta$ is the uniform quantization step

  - This noise is signal dependent and the signal-to-noise (SNR) is given by

$$SNR_{dB} \approx 10 \cdot \log\left(\sigma_x^2\right) + 6 \cdot b$$ where $\sigma_x^2$ is the signal variance and $b$ the number of bits

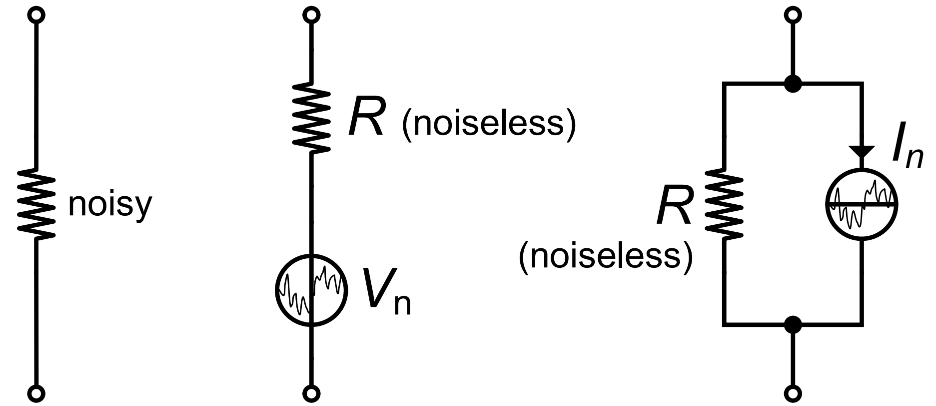
# Summary

- **Thermal noise** is due to fluctuation of drift velocity within a conductor and has a PSD **proportional to temperature** and to the **resistance** $S_{th} = 4kT \cdot R$. It is independent of the current flowing through the resistor
- The variance of the noise voltage on the capacitor of a 1st-order low-pass filter is given by $kT/C$, which is **independent of the resistance** and **inversely proportional to $C$**
- **Shot noise** is due to carriers randomly crossing a potential barrier. Its PSD is proportional to the average current $S_{sh} = 2q \cdot I$
- **Flicker noise** is due to fluctuation of the number of carrier (Mc Worther model) and/or of the mobility (Hooge model) and has a PSD inversely proportional to frequency
- The flicker noise PSD of MOS transistor is inversely proportional to the **gate area**
- Integrated white noise (Wiener process) has an ACF and a variance which grows with time

# Outline

- Introduction
- Random signals and noise
- Main noise sources of circuit components
- **Noise models of basic components**
- Noise calculation in continuous-time (CT) circuits
- Summary

# Device Noise – Resistor Noise

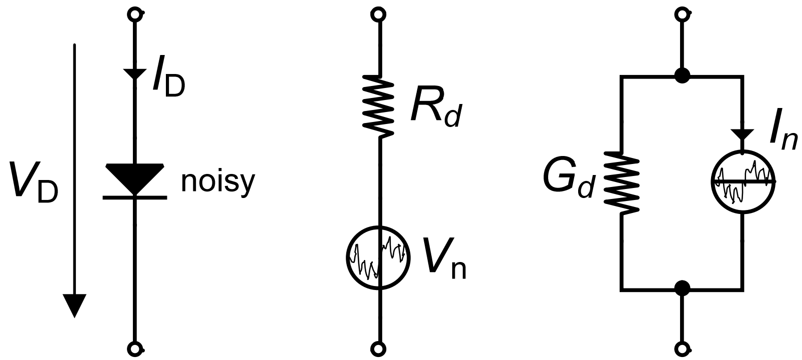



$$S_{V_n} = 4kT \cdot R \qquad S_{I_n} = \frac{4kT}{R}$$

- **Thermal noise** with $k$ being the Boltzman constant ($k = 1.38 \times 10^{-23}\, J/K$) and $T$ the absolute temperature in K
- Contrary to shot noise, the noise PSD of thermal noise is **independent of the current** flowing through the resistor or voltage across it, it only depends on the resistance $R$ (conductance $G = 1/R$)

# Device Noise – Diode Noise

$$I_D = I_S \cdot \left[ e^{\frac{V_D}{nU_T}} - 1 \right]$$



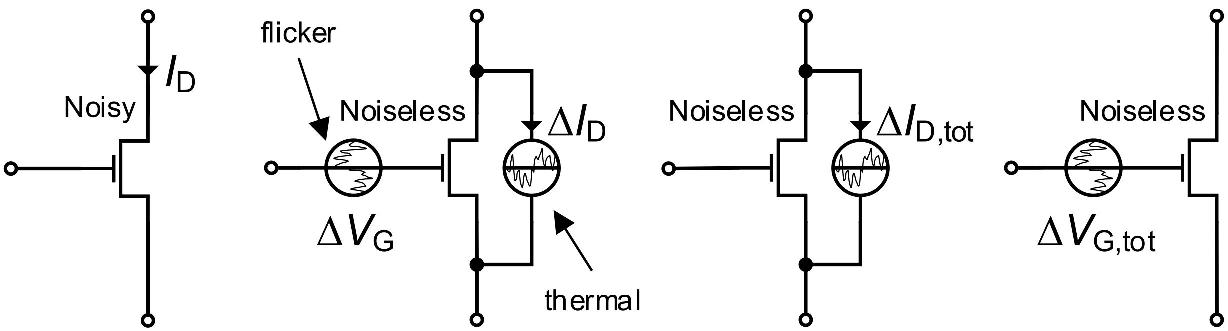

$$G_d = \frac{1}{R_d} \triangleq \frac{dI_D}{dV_D} = \frac{I_D}{nU_T}$$

$$S_{V_n} = 4kT \cdot \frac{R_d}{2}$$

$$S_{I_n} = 4kT \cdot \frac{G_d}{2} = 2q \cdot I_D$$

- Shot noise
- $G_d$ is the small-signal conductance
- Noise PSD **proportional to average current** flowing through the diode

# Device Noise – MOSFET (in saturation)




- **Thermal** noise (at the drain):

$$S_{\Delta I_D^2} = 4kT \cdot G_n \text{ with } G_n = \gamma_n \cdot G_m \text{ and } \gamma_n \cong 1 \text{ for long-channel}$$

- **Flicker** noise (at the gate):

$$S_{\Delta V_G^2}(f) = \frac{KF}{C_{ox}^{\alpha} \cdot W \cdot L \cdot f} = 4kT \cdot R_n(f) \text{ with } R_n(f) = \frac{\rho}{W \cdot L \cdot f} \text{ and } \rho = \frac{KF}{4kT \cdot C_{ox}^{\alpha}}$$

- **Total** noise at the drain:

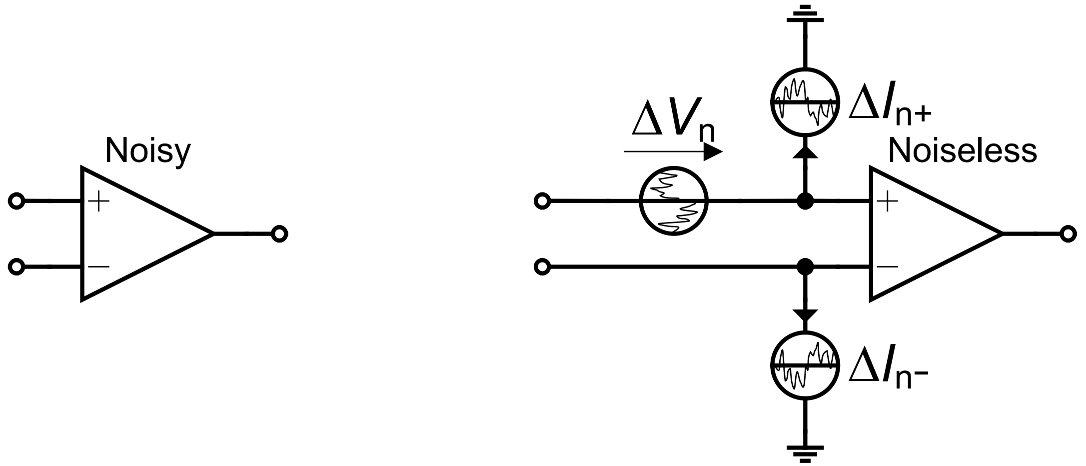
$$S_{\Delta I_{D,tot}^2}(f) = S_{\Delta I_D^2} + G_m^2 \cdot S_{\Delta V_G^2}(f) = 4kT \cdot G_{n,tot}(f) \text{ with } G_{n,tot}(f) = \gamma_n \cdot G_m + G_m^2 \cdot \frac{\rho}{W \cdot L \cdot f}$$

- **Total** input referred noise (at the gate):

$$S_{\Delta V_{G,tot}^2}(f) = 4kT \cdot R_{n,tot}(f) \text{ with } R_{n,tot}(f) = \frac{\gamma_n}{G_m} + \frac{\rho}{W \cdot L \cdot f}$$

C. C. Enz and E. A. Vittoz, *Charge-based MOS Transistor – The EKV model for low-power and RF IC design*, Wiley, 2006.

# Device Noise – OPAMP

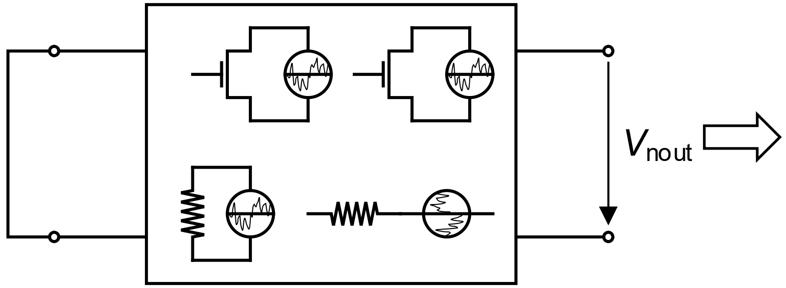



- Requires 3 noise sources
- All 3 noise sources are needed to have a model independent of the generator impedance
- Current noise sources can be ignored for MOS input stage

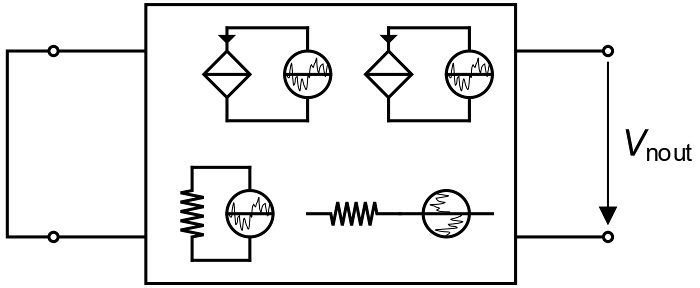
# Outline

- Introduction
- Random signals and noise
- Main noise sources of circuit components
- Noise models of basic components
- **Noise calculation in continuous-time (CT) circuits**
- Summary

# Noise Analysis in CT Circuits – Small-signal Circuit




Circuit including all noise sources



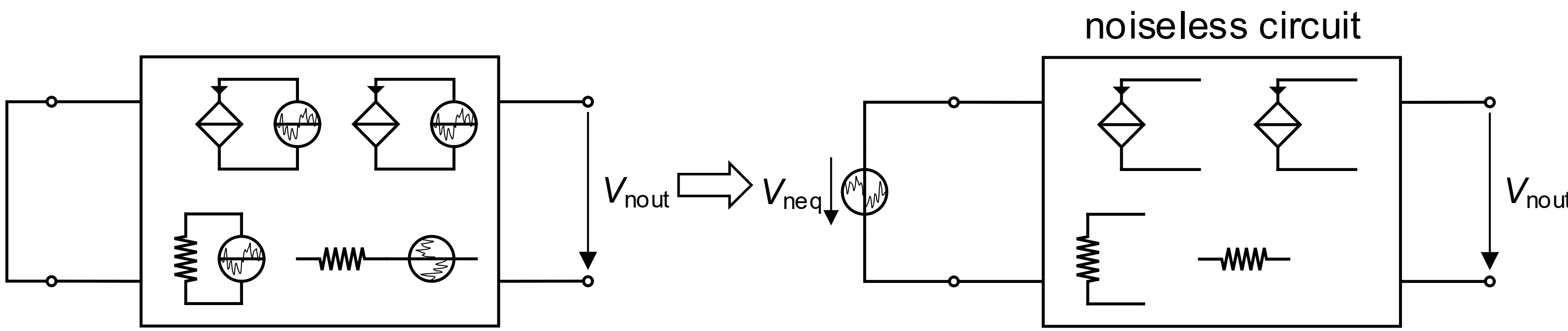

Small-signal equivalent circuit including all noise sources

- Since noise is a small perturbation ($\ll U_T$ at room temperature), the circuit can be **linearized** around an well defined operating point
- The output noise voltage PSD is calculated from the small-signal circuit as

$$S_{nout} = \sum_{k=1}^{K} |H_k(f)|^2 \cdot S_{nk}(f)$$

- where it has been assumed that all the $K$ noise sources are **uncorrelated**
- $H_k$ are the transfer functions from each noise source $k$ having a PSD $S_{nk}$ to the output

# Noise Analysis in CT Circuits – Input-referred Noise




Small-signal equivalent circuit including all noise sources

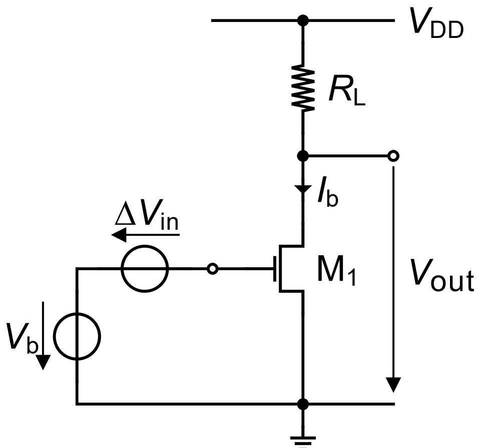
Noiseless circuit including equivalent input-referred noise source

- The noisy circuit can then modeled by a **noiseless circuit** with a single **input-referred noise** voltage source $V_{neq}$ producing the same noise at the output as that of the noisy circuit having the input short-circuited

$$S_{V_{neq}}(f) = \frac{S_{nout}(f)}{|A(f)|^2}$$

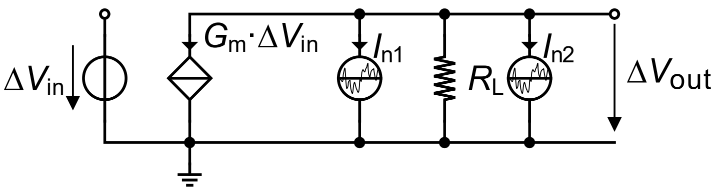
- where $A(f)$ is the transfer function from the input to the output

# Example – The CS Gain Stage




$$A_v \triangleq \frac{\Delta V_{out}}{\Delta V_{in}} = -G_m \cdot R_L$$

Small-signal circuit including noise sources of $M_1$ and $R_L$




- The output noise PSD is given by

$$S_{nout} = R_L^2 \cdot \left(S_{I_{n1}} + S_{I_{n2}}\right)$$

- where $S_{I_{n1}} = 4\,kT \cdot G_{n1}$ and $S_{I_{n2}} = 4\,kT \cdot G_{n2}$
- with $G_{n1} = \gamma_n \cdot G_m + G_m^2 \cdot \frac{\rho}{W \cdot L \cdot f}$ and $G_{n2} = 1/R_L$
- The input-referred noise PSD is then given by

$$S_{nin} = \frac{S_{nout}}{|A_v|^2} = 4kT \cdot R_{nin}$$

- with $R_{nin} = (G_{n1} + G_{n2})/G_m^2$

# Example – The CS Gain Stage

- The input-referred **thermal noise resistance** is given by

$$R_{nin,th} = \frac{\gamma_n}{G_m} \cdot (1 + \eta_{th})$$

- where the $\eta_{th}$ factor accounts for the contribution to the input-referred thermal noise of the load resistance $R_L$ normalized to that of the differential pair

$$\eta_{th} = \frac{1}{\gamma_n \cdot G_m R_L}$$

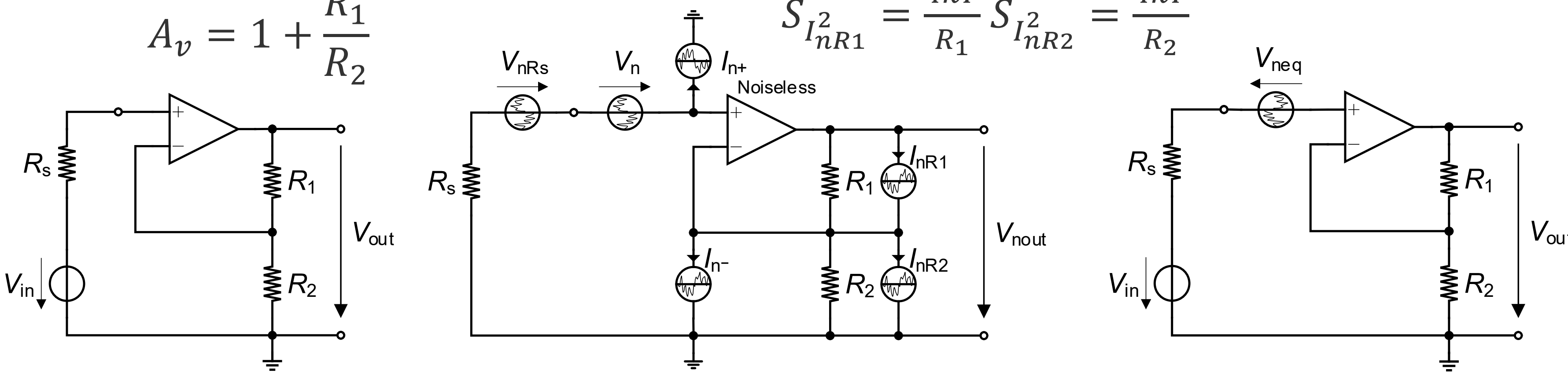
- We see that the higher the voltage gain $G_m R_L$, the lower the contribution of the load resistance to the input-referred thermal noise
- The input-referred flicker noise resistance is given by

$$R_{nin,fl} = \frac{\rho}{W \cdot L \cdot f}$$

- which only includes the contribution of the transistor M1 since the load resistance is assumed to only generate thermal noise

# Example – Noise of the Non-inverting Amplifier

$$A_v = 1 + \frac{R_1}{R_2} \qquad S_{I^2_{nR1}} = \frac{4kT}{R_1} \quad S_{I^2_{nR2}} = \frac{4kT}{R_2}$$

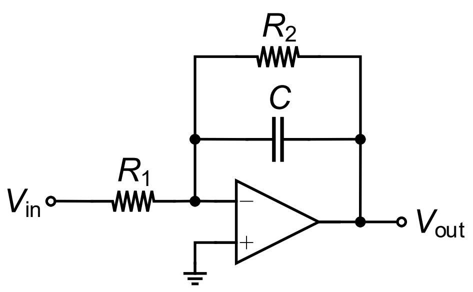



- The non-inverting amplifier has 6 noise source as shown on the middle schematic
- Assuming an ideal amplifier (except the noise) the equivalent input-referred noise voltage PSD is given by

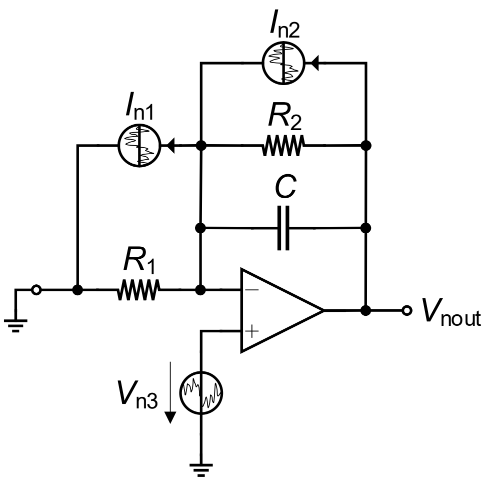
$$S_{V_{neq}} = S_{V_n} + 4kT \cdot (R_s + R_{12}) + R_s^2 \cdot S_{I_{n+}} + R_{12}^2 \cdot S_{I_{n-}} \text{ with } R_{12} = R_1 \parallel R_2$$

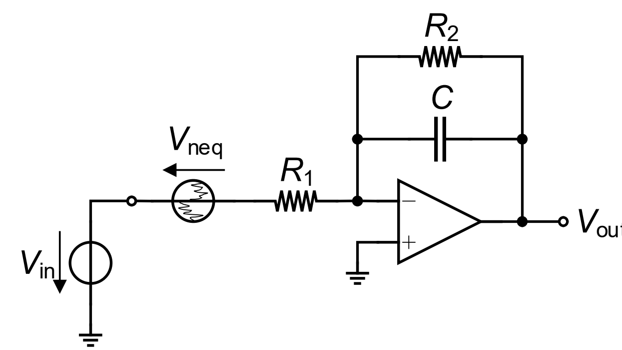
- Contribution of $R_1$ and $R_2$ equivalent to an additional resistance $R_{12}$ in series with $R_s$ and corresponding to $R_1 \parallel R_2$
- The OPAMP noise current sources start to dominate at high source impedance

# Example – 1st-order Low-pass Filter




$$R_1 = R_2 = R$$




- Assuming an ideal OPAMP (except the noise where we consider a high input impedance OPAMP and hence the current noise sources can be neglected) and $R_1 = R_2 = R$, the signal transfer function is given by

$$H(s) \triangleq \frac{V_{out}}{V_{in}} = -\frac{1}{1+\frac{s}{\omega_c}} \text{ with } \omega_c = \frac{1}{R \cdot C} = \frac{1}{\tau}$$

- The output noise voltage is given by

$$V_{nout} = R_{m12} \cdot (I_{n1} - I_{n2}) + H_{n3} \cdot V_{n3}$$

- with

$$R_{m12} = \frac{R}{1+\frac{s}{\omega_c}} \text{ and } H_{n3} = \frac{2+\frac{s}{\omega_c}}{1+\frac{s}{\omega_c}}$$

# 1st-order CT LPF – Output- and input-referred PSD

- The output noise voltage PSD is then given by

$$S_{V_{nout}}(f) = |R_{m12}(f)|^2 \cdot \left(S_{I_{n1}} + S_{I_{n2}}\right) + |H_{n3}(f)|^2 \cdot S_{V_{n3}}$$

- where the noise sources PSD are given by

$$S_{I_{n1}} = S_{I_{n2}} = \frac{4kT}{R} \text{ and } S_{V_{n3}} = 4kT \cdot \frac{\gamma}{G_m} \cdot \left(1 + \frac{f_k}{|f|}\right)$$

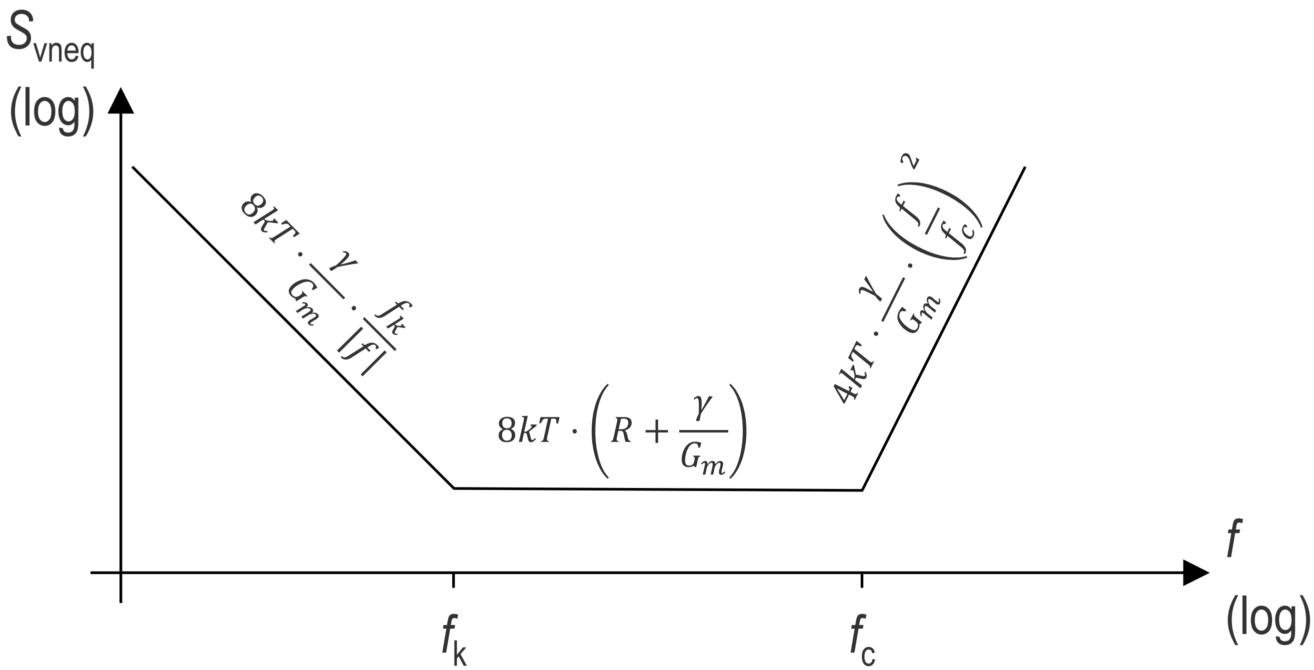
- The output noise voltage PSD is then given by

$$S_{V_{nout}}(f) = \frac{8kT \cdot R}{1 + \left(\frac{f}{f_c}\right)^2} + \frac{2 + \left(\frac{f}{f_c}\right)^2}{1 + \left(\frac{f}{f_c}\right)^2} \cdot 4kT \cdot \frac{\gamma}{G_m} \cdot \left(1 + \frac{f_k}{|f|}\right)$$

- which can be referred to the input as

$$S_{V_{neq}}(f) = 8kT \cdot R + \left[2 + \left(\frac{f}{f_c}\right)^2\right] \cdot 4kT \cdot \frac{\gamma}{G_m} \cdot \left(1 + \frac{f_k}{|f|}\right)$$

# 1st-order Low-pass Filter – Input Noise Voltage PSD




$$S_{V_{neq}}(f) = 8kT \cdot R + \left(2 + \left(\frac{f}{f_c}\right)^2\right) \cdot 4kT \cdot \frac{\gamma}{G_m} \cdot \left(1 + \frac{f_k}{|f|}\right)$$

$$= 8kT \cdot \frac{\gamma}{G_m} \cdot \frac{f_k}{|f|} + 8kT \cdot \left(R + \frac{\gamma}{G_m}\right) + 4kT \cdot \frac{\gamma}{G_m} \cdot \frac{f}{f_c} \cdot \left(\frac{f_k}{|f|} + \frac{f}{f_c}\right)$$

$$\cong 8kT \cdot \frac{\gamma}{G_m} \cdot \frac{f_k}{|f|} + 8kT \cdot \left(R + \frac{\gamma}{G_m}\right) + 4kT \cdot \frac{\gamma}{G_m} \cdot \left(\frac{f}{f_c}\right)^2 \text{ since } f_k < f_c$$

# 1st-order CT LPF – Output Noise Voltage Variance

- The variance of the output noise voltage can be calculated by integrating the output noise voltage PSD over frequency
- The contribution of the resistor $R_1$ and $R_2$ are 1st-order low-pass filtered
- The corresponding noise bandwidth is then simply

$$B_{n12} = \frac{\pi}{2} \cdot f_c = \frac{\pi}{2} \cdot \frac{1}{2\pi\tau} = \frac{1}{4\tau} = \frac{1}{4RC}$$

- The variance of the output noise voltage due to $R_1$ and $R_2$ is then simply given by

$$V_{nout}^2\Big|_{R_1,R_2} = \frac{8kT}{R} \cdot |R_{m12}(0)|^2 \cdot B_{n12} = \frac{8kT}{R} \cdot R^2 \cdot \frac{1}{4RC} = \frac{2kT}{C}$$

- The contribution of $V_{n3}$ to the output noise voltage PSD is not frequency bounded and hence its contribution to the output noise voltage variance is infinite
- In order to evaluate the output noise voltage variance due to $V_{n3}$, we need to account for the amplifier finite bandwidth

# 1st-order CT LPF – Amplifier Noise Transfer Function

- If the amplifier gain is assumed to be $A$, then the transfer function $H_{n3}$ becomes

$$H_{n3}(s) = \frac{2 + \frac{s}{\omega_c}}{1 + \frac{2}{A} + \left(1 + \frac{1}{A}\right)\frac{s}{\omega_c}}$$

- Note that for $A \to \infty$, we recover the earlier result
- If $A$ is simply approximated by $A = \omega_u / s$ where $\omega_u = G_m \,/\, C_c$ is the **gain-bandwidth product** with $C_c$ being the **compensation capacitance**, $H_{n3}$ becomes

$$H_{n3}(s) = \frac{2 + \frac{s}{\omega_c}}{1 + \left(2 + \frac{\omega_u}{\omega_c}\right)\frac{s}{\omega_c} + \frac{s^2}{\omega_u \omega_c}} = H_{n3}(0) \cdot \frac{1 + \frac{s}{\omega_z}}{1 + \frac{s}{\omega_0 \cdot Q} + \left(\frac{s}{\omega_0}\right)^2}$$

- with $H_{n3}(0) = 2$, $\omega_z = 2\omega_c$, $\omega_0^2 = \omega_c \cdot \omega_u$, $\omega_0 \cdot Q = \omega_u / (2 + \omega_u / \omega_c)$
- The amplifier noise is now bounded by the frequency dependence of the amplifier gain

# 1st-order CT LPF – Total Output Noise Voltage Variance

- From the table, it can be shown that the noise bandwidth corresponding to the 2nd-order transfer function $H_{n3}$ is given by

$$B_{n3} = \frac{\omega_0 \cdot Q}{4} \cdot \left[1 + \left(\frac{\omega_0}{\omega_z}\right)^2\right] = \frac{\omega_u}{16} \cdot \frac{4+\omega_u/\omega_c}{2+\omega_u/\omega_c} \cong \frac{\omega_u}{16} \text{ for } \omega_c \ll \omega_u$$

- The output noise voltage variance due to $V_{n3}$ is then given by

$$V_{nout}^2\Big|_{V_{n3}} = 4kT \cdot \frac{\gamma}{G_m} \cdot |H_{n3}(0)|^2 \cdot B_{n3} \cong 4kT \cdot \frac{\gamma}{G_m} \cdot 4 \cdot \frac{\omega_u}{16} = \frac{\gamma kT}{C_c}$$

- The total output noise voltage variance is then given by

$$V_{nout}^2 \cong \frac{2kT}{C} + \frac{\gamma kT}{C_c} = \frac{kT}{C_c} \cdot \left(\gamma + \frac{C_c}{C}\right) \cong \frac{\gamma kT}{C_c} \text{ for } C_c \ll C$$

- Since the voltage gain in the passband is unity, this also corresponds to the total input-referred noise voltage variance
- Note that if $C_c \ll C$, the noise voltage variance at the output is then mainly determined by the OPAMP thermal noise excess factor $\gamma$ and its compensation capacitor $C_c$ !

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# Summary

- Noise is a random source of perturbation inherent to any passive and active electronic device
- Because it is random it can be reduced by different techniques (such as filtering), but can never be eliminated
- Thermal noise has a PSD given by $S_{th} = 4kT \cdot R$, which is independent of the current flowing through the resistor
- The variance of the noise voltage on the capacitor of a 1st-order low-pass filter is given by $kT/C$
- Shot noise has a PSD proportional to the average current $S_{sh} = 2q \cdot I$
- The gate-referred thermal noise PSD of a MOS transistor is given by $S_{\Delta V_G^2} = 4kT \cdot \gamma_{nD}/G_m$
- The gate-referred flicker noise PSD of a MOS transistor is almost biased independent and given by$S_{\Delta V_G^2} = KF/(C_{ox}^{\alpha} \cdot W \cdot L \cdot f)$